Fourier transform methods for pathwise covariance estimation in the presence of jumps

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(based on joint work with David Skovmand and Josef Teichmann)

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- In particular, affine models can incorporate these ingredients and constitute a class which is flexible enough to capture the various stylized facts implied by market data.

Calibration problem:

But how can we determine in such models the parameters associated with the covariance process and the jump structure under some risk neutral measure?

Usual calibration approach

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- derivatives' data, usually prices of European call or put options with different maturities and strikes,
- models which allow for (semi-)analytic pricing formulas for European options,
- an optimization procedure, which is usually based on a least square criterium to minimize the distance between the model and market prices (or implied volatility, etc.).

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- Question:
  - Is there a reasonable way to use both sources of information simultaneously to estimate the model parameters?

Calibration concept - example of the Bates model

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\begin{aligned}
Y_t &= y + \int_0^t (\mu - \frac{1}{2} X_s - \int (e^{\xi} - 1 - \xi) m(d\xi)) ds + \int_0^t \sqrt{X_s} dZ_s \\
&\quad + \int_0^t \int \xi (\mu^Y(d\xi, ds) - m(d\xi) ds), \\
X_t &= x + \int_0^t \kappa(\theta - X_s) ds + \int_0^t \sigma \sqrt{X_s} dW_s.
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  - Z, W: one-dimensional correlated $\mathbb{P}$-Brownian motions, satisfying $d\langle Z, W\rangle_t = \rho dt$ for $\rho \in [-1, 1]$
  - μ, κ, θ, σ some (admissible) parameters
  - $m^Y(d\xi, dt)$: Poisson random measure with compensator $m(d\xi)dt$, where $m(d\xi)$ denotes some Lévy measure

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- Having fixed σ and ρ from time series observations, we obtain a family of equivalent measures on the canonical space of càdlàg paths $\mathcal{L} := \left(L^{\mu,\kappa,\theta,m(d\xi)}\right) = (Y^{\mu,\kappa,\theta,m(d\xi)}{}_*\mathbb{P})$. We here only consider equivalent measure changes which stay in the setting of the Bates model.

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- Since we aim to determine the model under some equivalent martingale measure $\mathbb{Q}$, we only select those $L^{\mu,\kappa,\theta,m(d\xi)}$ from $\mathcal{L}$ under which $e^{Y^{\mu,\kappa,\theta,m(d\xi)}}$ is a martingale. This necessarily means that $\mu = 0$ while all other parameters remain to be determined, necessarily from (a time series of) option prices.

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Parameters which remain invariant under equivalent measure changes can and should be determined from the time series of asset prices. In particular one should be able to identify those invariant parameters from "perfect" observations, if the model is correct.

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Paradigm 2:

Parameters which determine the particular risk neutral measure implied by derivatives' data have to be inferred from option prices and their term structures.

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- in a multivariate setting,
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This involves a two step procedure where we first need to recover the realized path of the instantaneous covariance from which we can then estimate the vol of vol and the correlation.

Literature

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  - Consistency and central limit theorems have first been proved for rather general estimators in a diffusion setting (Barndorff-Nielsen et al. (2006), Thesis of Podolskij (2006)) and have then been partly extended to processes with jumps (Woerner (2004), Barndorff-Nielsen et al. (2007), Andersen et al. (2008), Podolskij and Vetter (2009).)

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  - Recently Tauchen and Todorov (2011) introduced another jump robust estimator based on the realized Laplace transform of volatility.

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  - A method which avoids the issue of numerical differentiation is the Fourier method introduced by Malliavin and Mancino (2002, 2009), which however only works in a pure diffusion setting.
  - A two step procedure for vol of vol estimation has been considered by Barucci and Mancino (2010) in a pure diffusion setting and without proving a central limit theorem for the estimator.

Goal and procedure

1. Combine jump robust estimators (already considered for integrated covariance estimation or new specifications) with instantaneous covariance estimation based on Fourier methods:
   Reconstruct non-parametrically from the observations of the log-price processes the trajectories of the instantaneous covariance process.

$$\underbrace{(Y_{t_0}, Y_{t_1} \dots, Y_T)}_{\text{observed log-price}} \quad \xrightarrow{(1)} \quad \underbrace{(\widehat{X}_t)_{t \in [0,T]}}_{\text{estimator for the realized path of } X}$$

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2. Further develop the two step procedure, that is, use the path of $\widehat{X}$ to compute an estimator for the integrated covariance of the covariance process.

$$\underbrace{(\widehat{X}_t)_{t\in[0,T]}}_{\text{estimator for the realized path of } X} \xrightarrow{(2)} \underbrace{\int_0^T \widehat{Q}_s ds}_{\text{estimator for } \langle X_T^c, X_T^c\rangle}$$

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3. Determine under some parametric specification e.g. affine S_d^+-valued diffusion process, those parameters which remain invariant under equivalent measure changes.

Setting for multi-asset models

- We consider a d-dimensional (discounted) nonnegative asset price process $(S_t)_{t\geq 0}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

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- $(Y_t)_{t\geq 0}$ denotes the d-dimensional logarithmic (discounted) price process.
- Due to no-arbitrage considerations, S and thus also Y are supposed to be semimartingales.

Assumption on the log-price process

Assumption (H)

Structural assumption on Y: Y is an Itô-semimartingale of the form

$$Y_t = y + \int_0^t b_s ds + \int_0^t \sqrt{X_{s-}} dZ_s + \int_0^t \int_{\{\|\xi\| \leq 1\}} \xi(\mu^Y(d\xi, ds) - K_{\omega,s}(d\xi)ds) + \int_0^t \int_{\{\|\xi\| > 1\}} \xi \mu^Y(d\xi, ds),$$

- Z: *d-dimensional Brownian motion,*
- b: *$\mathbb{R}^d$-valued locally bounded predictable process,*
- X: *càdlàg process taking values in the cone of positive definite matrices S_d^+,*
- μ^Y: *random measure associated with the jumps of Y with compensator $K_{\omega,t}(d\xi)dt$.*

Assumptions on the instantaneous covariance process

Assumption (H1)

Hypothesis H holds and an additional structural assumption on the instantaneous covariance process X: X is an Itô-semimartingale of the form

$$X_t = x + \int_0^t b_s^X ds + \sum_{q=1}^{p} \int_0^t Q_{s-}^q dB_{s,q} + \int_0^t \delta(\xi, s-)(\mu^X(d\xi, ds) - F(d\xi)ds),$$

- B: *q-dimensional Brownian motion, which can be correlated with Z, the Brownian motion driving the log-price process.*
- b^X: *$\mathbb{R}^d$-valued locally bounded predictable process,*
- $(Q^q)_{q\in\{1,\dots,p\}}$: *càdlàg process taking values in S_d,*
- μ^X: *Poisson random measure on $S_d \times [0,\infty)$ independent of Z and B with compensator $F(d\xi)dt$,*
- $\delta(\omega,\xi,s)$: *a map $\Omega \times S_d \times [0,\infty) \to S_d$ satisfying certain technical conditions.*

Pathwise estimation of X – Fourier methodology

The main idea of the Fourier method for non-parametric pathwise covariance estimation can be described by the following steps:

1. Recover from (high frequency) observations of Y the Fourier coefficients of the components of the path $t \to \rho(X_t(\omega))$ for some continuous invertible function $\rho : S_d \to S_d$. In other words, find an estimator for

$$\mathcal{F}(\rho(X))(k) := \frac{1}{T}\int_0^T \rho(X_t) e^{-\mathrm{i}\frac{2\pi}{T}kt} dt.$$

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2. Use Fourier-Féjer inversion to reconstruct the path of $t \to \rho(X_t)$. Indeed, by Féjer's theorem

$$\sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) \mathcal{F}(\rho(X))(k) e^{\mathrm{i}\frac{2\pi}{T}kt}$$

converges uniformly to $\rho(X_t)$ on $[0, T]$ if $t \to X_t$ is continuous. If X has càdlàg paths, then the limit is given by $\frac{\rho(X_t)+\rho(X_{t-})}{2}$.

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3. Invert the function ρ to obtain the paths of the components of X.

How to obtain estimators for the Fourier coefficients

- Realize that the only difference with respect to estimators for integrated (functions of the) covariance are the terms $e^{-\mathrm{i}\frac{2\pi}{T}kt}$ in the integral for the Fourier coefficients.

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  - certain power variation estimators considered by Barndorff-Nielsen et al.
  - estimators for the realized Laplace transform of volatility introduced by Tauchen and Todorov and
  - other new jump robust specifications.

Review of integrated realized covariance estimators

- Time grids: For simplicity we suppose that the time grids of observations for all components of Y in $[0, T]$ are equal and equidistant, i.e.,

$$t_m^n = \frac{m}{n}, \quad m = 0, \ldots, \lfloor nT \rfloor$$

and the increments of a process Z with respect to the above time grid are denoted by $\Delta_m^n Z = Z_{t_m^n} - Z_{t_{m-1}^n}$.

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- The integrated variance estimators we consider are of the form

$$V(Y, g, 0)_T^n := \frac{1}{n} \sum_{m=1}^{\lfloor nT \rfloor} g(\sqrt{n} \Delta_m^n Y),$$

  for some function $g : \mathbb{R}^d \to S_d$.

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  - power variation estimators:
    $g : \mathbb{R}^d \to S_d, \quad (x_1, \ldots, x_d)^\top \mapsto (|x_i x_j|^{\frac{r}{2}})_{i,j \in \{1,\ldots,d\}}$

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- Time grids: For simplicity we suppose that the time grids of observations for all components of Y in $[0, T]$ are equal and equidistant, i.e.,

$$t_m^n = \frac{m}{n}, \quad m = 0, \dots, \lfloor nT \rfloor$$

and the increments of a process Z with respect to the above time grid are denoted by $\Delta_m^n Z = Z_{t_m^n} - Z_{t_{m-1}^n}$.

- The integrated variance estimators we consider are of the form

$$V(Y, g, 0)_T^n := \frac{1}{n} \sum_{m=1}^{\lfloor nT \rfloor} g(\sqrt{n} \Delta_m^n Y),$$

for some function $g : \mathbb{R}^d \to S_d$.

- Examples for g:
  - power variation estimators:
    $g : \mathbb{R}^d \to S_d, \quad (x_1, \dots, x_d)^\top \mapsto (|x_i x_j|^{\frac{r}{2}})_{i,j \in \{1,\dots,d\}}$
  - multivariate version of the Tauchen-Todorov estimator
    $g : \mathbb{R}^d \to S_d, \quad (x_1, \dots, x_d)^\top \mapsto (\cos(x_i + x_j))_{i,j \in \{1,\dots,d\}}$

Assumptions on the function g

In order to study asymptotic properties, the function g satisfies either

Assumption (J)

The function g is continuous with at most polynomial growth.

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Assumption (K)

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or

Assumption (K')

The function g is even and satisfies some technical conditions which are satisfied by $(x_1, \ldots, x_d)^\top \mapsto (|x_i x_j|^{\frac{r}{2}})_{i,j \in \{1,\ldots,d\}}$ for $r < 1$. (compare Barndorff-Nielsen et al. (2006)). Additionally, the instantaneous covariance process X is everywhere invertible.

Theorem (Barndorff-Nielsen et al. (2006))

Let Y be a continuous semimartingale satisfying assumption (H) and suppose that g satisfies (J), then

$$V(Y,g,0)^n_T \xrightarrow{\mathbb{P}} \int_0^T \rho_g(X_s)ds, ,$$

where $\rho_g(X) = \mathbb{E}\left[g(U)\right],\ U \sim \mathcal{N}(0,X)$. Under assumption (H_1) and (K) or (K')

$$\sqrt{n}\left(V(Y,g,0)^n_T - \int_0^T \rho_g(X_s)ds\right)$$

converges stably in law to an $\mathcal{F}$-conditional Gaussian random variable defined on an extension of the original probability space with mean 0 and covariance

$$\int_0^T \left(\rho_{g_{ij}g_{kl}}(X_s) - \rho_{g_{ij}}\rho_{g_{kl}}(X_s)\right) ds.$$

Remark

- *Stable convergence in law for a sequence of random variables (U_n) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a limit U (defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$) means that, for any bounded continuous function f and any bounded $\mathcal{F}$-measurable random variable V, we have*

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- *In these examples the function $\rho_g(x)$ corresponds to $\rho_{(x \mapsto |x|^r)}(x) = |x|^{\frac{r}{2}} \mathbb{E}\left[|u|^r\right]$, $u \sim \mathcal{N}(0,1)$ and $\rho_{(x \mapsto \cos(x))}(x) = e^{-\frac{1}{\sqrt{2}}x}$.*

Integrated covariance estimators - inclusion of jumps

Proposition (C., D. Skovmand, J. Teichmann (2012))

Let Y be an Itô semimartingale satisfying assumption (H) and denote by D its diffusion part

$$D_t = y + \int_0^t b_s ds + \int_0^t \sqrt{X_{s-}} dZ_s.$$

Then under the same assumptions as in the previous theorem, the above consistency statement holds if for all $m \in \{1, \ldots, \lfloor nT \rfloor\}$

$$\lim_{n\to\infty} \mathbb{E}\left[\|g(\sqrt{n}\Delta_m^n Y) - g(\sqrt{n}\Delta_m^n D)\|\right] = 0$$

and the above CLT holds if for all $m \in \{1, \ldots, \lfloor nT \rfloor\}$

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Proposition (C., D. Skovmand, J. Teichmann (2012))

Let Y be an Itô semimartingale satisfying assumption (H) and suppose that g is globally α-Hölder continuous. Let $p \in [0,2)$ and assume that the jumps of Y satisfy $\mathbb{E}\left[\int_{\{\|\xi\| \leq 1\}} \|\xi\|^p K_{\omega,t}(d\xi)\right] < \infty$ and $\mathbb{E}\left[\int_{\{\|\xi\| > 1\}} \|\xi\|^\alpha K_{\omega,t}(d\xi)\right] < \infty$. Then

$$\lim_{n\to\infty} \mathbb{E}\left[\|g(\sqrt{n}\Delta_m^n Y) - g(\sqrt{n}\Delta_m^n D)\|\right] = 0$$

is always satisfied and

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holds if $p < 1$ and $\frac{p}{2-p} < \alpha < 1$.

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Proof.

$$\mathbb{E}\left[\|g(\sqrt{n}\Delta_m^n Y) - g(\sqrt{n}\Delta_m^n D)\|\right] \leq C\frac{1}{n^{\alpha\wedge\frac{\alpha}{p}-\frac{\alpha}{2}}}.$$

□

Integrated covariance estimators - new jump robust example

- Jump robust estimators for which a CLT of the above form holds true can thus be obtained by considering even functions which satisfy assumption (K) or $(K)'$ and additionally some global α-Hölder continuity.

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  - $g : \mathbb{R}^d \to S_d,\ x \mapsto (h_{A_{ij}})_{ij \in \{1,\dots,d\}} = \left(e^{-\frac{\langle x, A_{ij} x \rangle}{2}}\right)_{ij \in \{1,\dots,d\}}$ with $A_{ij} = e_i e_i^\top + e_j e_j^\top + e_i e_j^\top + e_j e_i^\top$

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  - $\rho_{g_{ij}}(X) = \mathbb{E}\left[h_{A_{ij}}(U)\right] = \frac{1}{\sqrt{X_{ii} + 2X_{ij} + X_{jj} + 1}}$

Assumptions on g and the jumps

Assumption (L(η))

For $\eta \geq 0$*, we have for all* $m \in \{1, \ldots, \lfloor nT \rfloor\}$

$$\lim_{n \to \infty} n^{\eta} \mathbb{E}\left[\| g(\sqrt{n} \Delta_m^n Y) - g(\sqrt{n} \Delta_m^n D) \| \right] = 0.$$

Estimators for the Fourier-coefficients

- The estimators for the Fourier coefficients of $t \mapsto \rho_g(X_t)$ are defined by

$$V(Y,g,k)^n_T = \frac{1}{n} \sum_{m=1}^{\lfloor nT \rfloor} e^{-\mathrm{i}\frac{2\pi}{T} k t^n_{m-1}} g(\sqrt{n}\Delta^n_m Y),$$

and we write

$$\begin{aligned} V(Y,g)^{n,K}_T &:= \left(V(Y,g,-K)^n_T, \dots, V(Y,g,0)^n_T, \dots, V(Y,g,K)^n_T\right)^\top \\ &= \frac{1}{n} \sum_{m=1}^{\lfloor nT \rfloor} \mathcal{B}^K(t^n_{m-1}) g(\sqrt{n}\Delta^n_m Y), \end{aligned}$$

where $\mathcal{B}^K(t) := (e^{-\mathrm{i}\frac{2\pi}{T}(-K)t}, \dots, 1, \dots, e^{-\mathrm{i}\frac{2\pi}{T}(K)t})^\top$.

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where $\mathcal{B}^K(t) := (e^{-\mathrm{i}\frac{2\pi}{T}(-K)t}, \dots, 1, \dots, e^{-\mathrm{i}\frac{2\pi}{T}(K)t})^\top$.

- In the sequel we denote the vector of Fourier coefficients of $t \mapsto f(t)$ by $\mathcal{F}^K(f) := (\mathcal{F}(f)(-K), \dots, \mathcal{F}(f)(0), \dots, \mathcal{F}(f)(K))^\top$

Asymptotic properties for the Fourier-coefficients estimator

Corollary (C., D. Skovmand, J. Teichmann)

Under the assumptions (H), (J) *and* $(L(0))$ *we have*

$$V(Y,g)_T^{n,K} \xrightarrow{\mathbb{P}} \int_0^T \mathcal{B}^K(s)\rho_g(X_s)ds = T\mathcal{F}^K(\rho_g(X))$$

as $n \to \infty$, *where* $\rho_g(X) = \mathbb{E}\left[g(U)\right], \ U \sim \mathcal{N}(0,X)$. *Under the assumption* (H_1) *and* (K) *or* (K') *and* $L(\frac{1}{2})$, *the* $\mathbb{R}^{(2K+1)d\times d}$*-valued random variable*

$$\sqrt{n}\left(V(Y,g)_T^{n,K} - T\mathcal{F}^K(\rho_g(X))ds\right)$$

converges for $n \to \infty$ *stably in law to an* $\mathcal{F}$*-Gaussian random variable defined on an extension of the original probability space with mean* 0 *and covariance*

$$\int_0^T \left(\rho_{g_{ij}g_{kl}}(X_s) - \rho_{g_{ij}}\rho_{g_{kl}}(X_s)\right) e^{-\mathrm{i}\frac{2\pi}{T}(p-q)s}ds.$$

Fourier estimator for the instantaneous covariance

- Using the estimators for Fourier coefficients, we can now define an estimator for a function of the instantaneous covariance, namely for $\rho_g(X_t)$, via Fourier-Féjer inversion:

$$\widehat{\rho_g(X)}_t^{n,N} = \frac{1}{T} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) e^{\mathrm{i}\frac{2\pi}{T}kt} V(Y, g, k)^n_T.$$

Consistency of the Fourier estimator for the instantaneous covariance

Theorem (C., D. Skovmand, J.Teichmann)

Let $\gamma > 1$ *and suppose that* $\lim \frac{n}{N^\gamma} = C$. *Under the assumptions* (H), (J) *and* $(L(0))$ *we have for every* $t \in [0, T]$

$$\widehat{\rho_g(X)}_t^{n,N} \xrightarrow{\mathbb{P}} \frac{\rho_g(X_{t-}) + \rho_g(X_t)}{2}.$$

as $n, N \to \infty$. *Convergence is uniform in* t, *if* $t \mapsto X_t$ *is continuous.*

CLT for the Fourier estimator for the spot volatility

Theorem (C., D. Skovmand, J.Teichmann)

Assume that the paths of X are almost surely Hölder continuous with exponent δ. Let $1 < \gamma < 2\delta + 1$ and suppose that $\lim \frac{n}{N^\gamma} = C$. Then under $(H1)$ and (K) or (K') and $L(\eta)$ with $\eta > \frac{\gamma - 1}{2\gamma}$, the $\mathbb{R}^{d \times d}$ random variable

$$n^{\frac{\gamma-1}{2\gamma}} \left(\widehat{\rho_g(X)}_t^{n,N} - \rho_g(X_t) \right) \tag{1}$$

converges for each $t \in [0, T]$ as $n, N \to \infty$ stably in law to a $\mathcal{F}$-conditional Gaussian random variable defined on an extension of the original probability space with mean 0 and finite non-zero covariance function given by

$$V_t^{ijkl} := \lim_{N \to \infty} \frac{1}{T^2} \int_0^T \left(\rho_{g_{ij} g_{kl}}(X_s) - \rho_{g_{ij}}(X_s) \rho_{g_{kl}}(X_s) \right) \frac{F_N^2 \left(\frac{2\pi}{T}(t - s) \right)}{N} ds,$$

where $F_N(t) = \sum_{-k=N}^{k=N} \left(1 - \frac{|k|}{N} \right) e^{ikt}$ denotes the Féjer kernel.

Remarks

Remark

- *Under* $(H1)$, *Hölder continuity of the trajectories of* X *with* $\delta < \frac{1}{2}$ *is satisfied if* X *has no jumps and if* $\mathbb{E}\left[\|b_u^X\|^p\right] < \infty$ *and* $\mathbb{E}\left[\|Q_u^q\|^p\right] < \infty$ *for all* $p \geq 1$ *and* $u \in [0, T]$.
- *In this case* $\gamma \in (1, 2)$. *The higher* γ *the better the convergence rate and it lies between* $(0, \frac{1}{4})$.
- $L(\eta)$ *with* $\eta > \frac{\gamma-1}{2\gamma}$ *is satisfied, if* g *is* α*-Hölder continuous and the jumps of* Y *satisfy* $\mathbb{E}\left[\int_{\{\|\xi\|\leq 1\}} \|\xi\|^p K_{\omega,t}(d\xi)\right] < \infty$ *and* $\mathbb{E}\left[\int_{\{\|\xi\|>1\}} \|\xi\|^\alpha K_{\omega,t}(d\xi)\right] < \infty$ *and* $\frac{2\alpha-2p}{p} > \frac{\gamma-1}{\gamma}$.

Corollary (C., D. Skovmand, J. Teichmann)

Let g be such that $\rho_g(x) : S_d \times S_d,\ x \mapsto \rho_g(x)$ *has a differentiable inverse and denote by*

$$\widehat{X}_t^{n,N} := \rho_g^{-1}\left(\widehat{\rho_g(X)}_t^{n,N}\right).$$

Then under the assumption of the above theorem

$$n^{\frac{\gamma-1}{2\gamma}}\left(\widehat{X}_t^{n,N} - X_t\right)$$

converges as $n, N \to \infty$ *for each* $t \in [0, T]$ *stably in law to a* $\mathcal{F}$*-conditional Gaussian random variable given by*

$$M_t = (\nabla \rho_g(X_t))^{-1} N_t$$

where N_t *denotes the limit of* (1).

Integrated covariance of the instantanous covariance process

- Having reconstructed the path of the instantaneous covariance, we can now proceed to the covariance estimation of the instantaneous covariance process and the determination of parameters under certain model specifications.

Integrated covariance of the instantanous covariance process

- Having reconstructed the path of the instantaneous covariance, we can now proceed to the covariance estimation of the instantaneous covariance process and the determination of parameters under certain model specifications.
- To do so we use the usual realized covariance estimator, where we plug in the reconstructed path of the instantaneous covariance process:

$$(V(\widehat{X}^{n,N},2)^m_T)_{ijkl} := \sum_{p=1}^{\lfloor mT \rfloor} |(\Delta_p^m \widehat{X}^{n,N})_{ij}||(\Delta_p^m \widehat{X}^{n,N})_{kl}|.$$

A CLT for the vol estimator of the covariance process

Theorem (C., D. Skovmand, J.Teichmann)

- *Let the conditions of the above theorem be in force and let g be such that* $x \mapsto \rho_g(x)$ *has a differentiable inverse.*
- *Suppose that the covariance of covariance process Q defined by* $Q_{s,ijkl} = \sum_{p,q} Q^p_{s,ij} Q^q_{s,kl}$ *satisfies* (H_1)*.*
- *Let* $\beta < \frac{\gamma-1}{2\gamma}$ *and assume that* $\lim \frac{m}{n^{2\beta}} = C$*.*

Then

$$n^{\beta} \left(V(\widehat{X}^{n,N}, 2)^m_T - \int_0^T Q_s ds \right)$$

converges as $n, N, m \to \infty$ *stably in law to a* $\mathcal{F}$*-conditional Gaussian random variable defined on an extension of the original probability space with* 0 *mean and covariance function given by*

$$F_{ijklmnpq} = \int_0^T (Q_{s,ijmn} Q_{s,klpq} + Q_{s,ijpq} Q_{s,klmn})\, ds.$$

Affine model specification

Affine stochastic covariance model

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Affine stochastic covariance model

- We consider affine models, namely multivariate Bates-type models, where X is described by a generalized Wishart process, i.e, an affine diffusion process of the form

$$dX_t = (b + MX_t + X_t M^\top)dt + \sqrt{X_t}dW_t\Sigma + \Sigma^\top dW_t^\top \sqrt{X_t},$$

where $M \in \mathbb{R}^{d\times d}$, $\Sigma \in S_d^+$, $b \in S_d^+$ s.t. $b - (d-1)\Sigma^\top\Sigma \in S_d^+$ and W a $d \times d$ matrix of Brownian motions correlated with the Brownian motion Z such that $Z = \sqrt{1-\rho^\top\rho}V + W\rho$ where $\rho \in [-1,1]^d$ s.t. $\rho^\top\rho \leq 1$,

Estimation of Σ

- In the case of affine models, we have

$$\langle Y_i^c, Y_j^c \rangle_T = \int_0^T X_{t,ij} dt$$

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- Estimating from the reconstructed path of X, the integrated covariance between the components of X, thus allows to estimate Σ via

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- Similarly an estimator for the correlation parameter ρ can be obtained.

Conclusion

- Our calibration concept is based on the
  - ...on the use of both time series and option price data,
  - ...on the fact that certain parameters such as the covariance of the covariance remain invariant under equivalent measure changes,
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  - ... on non-parametric instantaneous covariance estimators based on Fourier methods for which we establish central limit theorems.
- Work in progress, Outlook
  - Inclusion of jumps in the volatility process
  - Use of other estimator for the Fourier coefficients, e.g. the Bohr convolution estimator by Malliavin and Mancino
  - Application to real data

- Thank you for your attention!