Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

D MATH
Departement of Mathematics
Prof.em.dr. Freddy Delbaen

Dynamic Monetary Utility Functions and BSDE

ETH Zürich and UNI Zürich

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1 Monetary Utility Functions

$(\Omega, \mathcal{F}_1, \mathcal{F}_2, \mathbb{P})$ one period model with non-trivial initial information.

$u_1 : L^\infty(\mathcal{F}_2) \to L^\infty(\mathcal{F}_1)$

We call u_1 a monetary utility function if

1. $u_1(0) = 0, u_1(\xi) \geq 0$ if $\xi \geq 0$

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2. concavity: $0 \leq \lambda \leq 1$, λ $\mathcal{F}_1$ measurable then

$$u_1(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \lambda u_1(\xi_1) + (1-\lambda)u_1(\xi_2)$$

3. monetary: for $\eta \in L^\infty(\mathcal{F}_1)$, $u_1(\xi + \eta) = u_1(\xi) + \eta$
4. Fatou property: for $\xi_n \downarrow \xi$, $u_1(\xi_n) \downarrow u_1(\xi)$

Remark

The property with $\uparrow$ is stronger and implies a lot of other consequences, see weak compactness.

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Remark

We only define such “risk adjusted (e)valuation" on a space of bounded random variables. The extension to a space of unbounded random variables is non trivial and poses extra integrability problems. A further extension to the space of all random variables leads to contradictions or to situations where u must take the value $-\infty$.

Remark

The definition above implies that for $A \in \mathcal{F}_1$:

$$u_1(\mathbf{1}_A\xi) = \mathbf{1}_A\, u_1(\xi).$$

It also implies monotonicity:

if $\xi_1 \le \xi_2$ then $u_1(\xi_1) \le u_1(\xi_2)$.

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With u_1 we associate the convex set

$$\mathcal{A}_{1,2} = \{\xi \mid u_1(\xi) \geq 0\}.$$

This set is weak* (i.e. $\sigma(L^\infty, L^1)$) closed. It is convex and contains the cone, L^∞_+, of non-negative random variables.

$$u_1(\xi) = \text{ess.inf}\{\eta \mid \eta \in L^\infty(\mathcal{F}_1) \text{ and } \xi + \eta \in \mathcal{A}_{1,2}\}$$

This essential infimum is in fact a minimum.

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2 Extra Assumptions

$$u_1(\xi) \le \mathbb{E}[\xi \mid \mathcal{F}_1]$$

This assumption can be relaxed but it facilitates the presentation. As always with such assumptions there is a danger of neglecting essential difficulties.

Relevance: for each $\epsilon > 0$ and each $A \in \mathcal{F}_2$ with $\mathbb{P}[A] > 0$ we have $\mathbb{P}[u_1(-\epsilon \mathbf{1}_A) < 0] > 0$. This even implies that $u_1(-\epsilon \mathbf{1}_A) < 0$ on the set $\{\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1] > 0\}$.

3 Representation

Duality arguments, the Hahn-Banach theorem, show that on the set of equivalent probability measures, $\mathbf{P}^e$, there is a function

$$c_1 : \mathbf{P}^e \to L^0_+(\mathcal{F}_1; \overline{\mathbb{R}_+}), \text{ such that}$$

1. c_1 is convex, lower semi-continuous, $c_1(\mathbb{P}) = 0$,
2. $c_1(\mathbb{Q}) = \text{ess.sup}\{-\mathbb{E}_\mathbb{Q}[\xi] \mid \xi \in \mathcal{A}_{1,2}\}$
3. $u_1(\xi) = \text{ess.inf}\{\mathbb{E}_\mathbb{Q}[\xi \mid \mathcal{F}_1] + c_1(\mathbb{Q}) \mid \mathbb{Q} \in \mathbf{P}^e\}$

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4 Coherent Utilities

In case u_1 is also positively homogeneous (with $\mathcal{F}_1$ measurable coefficients), we call u_1 coherent. This is equivalent to saying that $\mathcal{A}_{1,2}$ is an $\mathcal{F}_1-$cone. In that case the function c_1 only take the values 0 and $+\infty$. The duality then reads

$$u_1(\xi) = \text{ess.inf}\{\mathbb{E}_{\mathbb{Q}}[\xi \mid \mathcal{F}_1] \mid \mathbb{Q} \in \mathcal{S}\},$$

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where $\mathcal{S}$ is a closed convex set of probability measures such that $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_1$. The relevance is implied by the assumption that $\mathbb{P} \in \mathcal{S}$. The subset of $\mathcal{S}$ consisting of equivalent measures is then dense in $\mathcal{S}$.

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5 Continuous Time

We make a big jump and move to continuous time in a bounded time interval $[0, T]$. The filtration $\mathcal{F}$ describes the revelation of uncertainty and we suppose that $\mathcal{F}$ satisfies the usual assumptions. We suppose that $\mathcal{F}_0$ is trivial, this is not a restriction as we can always extend the time interval to the left and we can add the functional $u_{-1}(\xi) = \mathbb{E}[u_0(\xi)]$. We do not give the notational details, as they are straightforward.

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We suppose that for each stopping time $0 \leq \sigma \leq T$ we have a monetary utility function:

$$u_\sigma \colon L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_\sigma).$$

Restriction to a pair of stopping times $0 \leq \sigma \leq \tau \leq T$ then gives the following objects

1. $\mathcal{A}_\sigma = \mathcal{A}_{\sigma,T} \quad \mathcal{A}_{\sigma,\tau} = \mathcal{A}_\sigma \cap L^\infty(\mathcal{F}_\tau)$
2. $c_{\sigma,\tau}(\mathbb{Q}) = \operatorname{ess.sup}\{-\mathbb{E}_\mathbb{Q}[\xi \mid \mathcal{F}_\sigma] \mid \xi \in \mathcal{A}_{\sigma,\tau}\}$

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3. for $\xi \in L^\infty(\mathcal{F}_\tau)$:

$$u_\sigma(\xi) = \text{ess.inf}\{\mathbb{E}_\mathbb{Q}[\xi \mid \mathcal{F}_\sigma] + c_{\sigma,\tau}(\mathbb{Q}) \mid \mathbb{Q} \in \mathbf{P}^e\}$$

The next paragraph deals with the evolution of u_t as a function of time.

6 Time Consistency

Essential for consistent decision making is the concept of “time consistency". As presented here it reflects the ideas of Koopmans (1960,1961) but is stated in probabilistic language. It might be too strong for some applications and could be replaced by weaker forms (see the Berlin School).

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We say that the family u_σ is time consistent if for each pair of stopping times, $0 \leq \sigma \leq \tau \leq T$:

$$u_\tau(\xi_1) \leq u_\tau(\xi_2) \text{ implies } u_\sigma(\xi_1) \leq u_\sigma(\xi_2).$$

Time consistency has a lot of nontrivial consequences. It also implies a structure on the penalty functions and allows to make a link with concepts of probability theory such as potentials, supermartingales, ...

The following are equivalent

1. Time consistency
2. For each pair of stopping times $0 \le \sigma \le \tau \le T$:
$$u_\sigma(\xi) = u_\sigma(u_\tau(\xi))$$
3. $u_0(\xi) = u_0(u_\tau(\xi))$
4. $\mathcal{A}_\sigma = \{\xi \mid \text{ for each } A \in \mathcal{F}_\sigma : \mathbf{1}_A \xi \in \mathcal{A}_0\}$

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5. for stopping times $\sigma \le \tau \le \nu$

$$\mathcal{A}_{\sigma,\nu} = \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,\nu}$$

6. same as above, for $\mathbb{Q} \in \mathbf{P}^e$

$$c_{\sigma,\nu}(\mathbb{Q}) = c_{\sigma,\tau}(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[c_{\tau,\nu}(\mathbb{Q}) \mid \mathcal{F}_\sigma]$$

For a time consistent utility function the utility u_0 completely determines the other functions. Given an arbitrary monetary utility u_0, there is at most one time consistent extension but as for instance tail expectation shows, the extension is not always possible. This again shows that time consistency is a very special property.

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7 Supermartingales and Regularity

Under time consistency we have extra properties. The most important is:

If $c_0(\mathbb{Q}) < \infty$, then the family $c_\sigma(\mathbb{Q})$ has the supermartingale property and is uniformly integrable (class D). It is called a potential of class D.

As a consequence we can show that there is a càdlàg version of $c_{t,T}(\mathbb{Q})$ reproducing the family $c_{\sigma,T}(\mathbb{Q})$. This implies that there is a càdlàg increasing adapted process $A^{\mathbb{Q}}$ such that

$$c_\sigma(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}[A_T^{\mathbb{Q}} - A_\sigma^{\mathbb{Q}} \mid \mathcal{F}_\sigma].$$

There is also a càdlàg version for the family u_σ and for each $\mathbb{Q}$ with $c_0(\mathbb{Q}) < \infty$:

$u_t(\xi) + A_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-submartingale. With the extra assumptions we get that $u_t(\xi)$ is a $\mathbb{P}$ submartingale and hence has a decomposition $du_t(\xi) = dA_t - dM_t$ where A is increasing and M is a martingale.

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8 Brownian Filtration

In the case of the filtration of a Brownian Motion, (W), we can go further and give more information on the increasing processes $A^{\mathbb{Q}}$. The main ingredient is the representation of densities using stochastic exponentials. If $\mathbb{Q} \sim \mathbb{P}$ then the density process is given by a stochastic exponential

$$\mathcal{E}(q \cdot W)_t = \exp\left(\int_0^t q_u \, dW_u - \frac{1}{2} \int_0^t q_u^2 \, du\right).$$

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It can be shown that there is a jointly measurable function

$$f\colon [0,T]\times\Omega\times\mathbb{R}\to\overline{\mathbb{R}_+}$$

with

1. for each (t,ω) the function is convex and lower semi continuous in q, $f(t,\omega,0)=0$
2. for each q, the function is predictable (optional)
3. $c_0(\mathbb{Q})=\mathbb{E}_{\mathbb{Q}}[\int_0^T f(t,.,q(.))\,dt]$, this formula can also be used for $\mathbb{Q}\ll\mathbb{P}$.

4. $c_\sigma(\mathbb{Q}) = \mathbb{E}_\mathbb{Q}\left[\int_\sigma^T f(u, ., q(.))\, du \mid \mathcal{F}_\sigma\right]$

5. $A_t^\mathbb{Q} = \int_0^t f(u, ., q(.))\, du$

6. $u_t(\xi) + \int_0^t f(u, ., q(.))\, du$ is a $\mathbb{Q}$ submartingale

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9 Relation with BSDE

We suppose that the function f only depends on q. The analysis can be extended for the more general case but it gets very technical. The Legendre transform of f is denoted by g

$$g(z) = \sup_q (q \cdot z - f(q)).$$

The function g can take the value $+\infty$. Now we need some extra functional analysis. The following are equivalent

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1. for each $k < \infty$, the set $\{\mathbb{Q} \mid c_0(\mathbb{Q}) \leq k\}$ is weakly compact in L^1

2. for each $\xi \in L^\infty$, there is a $\mathbb{Q}$ such that

$$u_0(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi] + c_0(\mathbb{Q}).$$

3. $\liminf_{q\to\infty} \frac{f(q)}{q^2} > 0$.

4. $\limsup_{z\to\infty} \frac{g(z)}{z^2} < \infty$.

The equivalence between 1 and 2 is true in general.

We have that for $\xi \in L^\infty$ and for a minimising measure $\mathbb{Q}_0$, the process

$$u_t(\xi) + A_t^{\mathbb{Q}_0},$$

is a $\mathbb{Q}_0-$martingale. This property remains true in general, i.e. in more general filtrations.

Under the circumstances above, we can show – using BMO theory – that $\mathbb{Q}_0 \sim \mathbb{P}$, $dA^{\mathbb{Q}_0} \ll dt$ and using the submartingale property for other measures we finally get that $dA^{\mathbb{Q}_0} = g(Z_t)\,dt$ and therefore also $du_t(\xi) = g(Z_t)\,dt - Z_t\,dW_t$. This reasoning heavily uses that g is subquadratic and that by convexity g' is linearly bounded. This means that we have shown the existence of a bounded solution for the equation

$$dY_t = g(Z_t)\,dt - Z_t\,dW_t \quad Y_T = \xi.$$

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Uniqueness of the bounded solution then follows using some well chosen sub-martingale.

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10 Example: Entropic Measures

We put $f(q) = \frac{1}{2}q^2$. Then $c_0(\mathbb{Q}) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$, the entropy.

$u_t(\xi) = -\log \mathbb{E}[\exp(-\xi) \mid \mathcal{F}_t]$

$\exp(-u_t(\xi))$ is a martingale

$g(z) = \frac{1}{2}z^2$

$u_t(\xi)$ is the solution of $dY_t = \frac{1}{2}Z_t^2\, dt - Z_t\, dW_t$.

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11 Superquadratic Case

The superquadratic case, i.e. $\liminf_{z\to\infty} \frac{g(z)}{z^2} = +\infty$ has some extra surprises:

1. For some ξ there is no bounded solution.
2. If for ξ there is a bounded solution, then necessarily there are infinitely many bounded solutions. They all satisfy $Y_0 \leq u_0(\xi)$.
3. The process $u_{.}(\xi)$ is not necessarily a solution even when for that ξ there is a bounded solution.