A Principal-Agent Problem for Emissions' reduction

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Contents

Framework

A firm (agent) produces emissions and the state (principal) wants to reduce them as much as possible. Problem: the state does not observe the effort (technology) undertaken by the firm, he can only give out incentives in order to control indirectly the level of emissions (that he observes). We consider two different kinds of incentives:

- "Negative" incentives, represented by a fee to pay at a given time T if emissions are too high
- "positive" incentives, in the form of continuous-time payments to the agent.

Related literature: Sannikov (2008), Cvitanic (2005), Williams (2008), Carmona (2011).

The model

Emissions process (in $\mathbb{R}_+$):

$$X_t = x + \int_0^t X_r l(k_r)dr + \int_0^t X_r \sigma dW_r^k \qquad (1.1)$$

where k represents the effort (technology used), taking values in $\mathbb{R}_+$ (where a higher value stands for a better technology), and l captures the effect of technology on the emissions ($l'(k) \leq 0$).
We can write equivalently

$$X_t = x + \int_0^t X_r \sigma dW_r^0 \qquad (1.2)$$

and recover (1.1) by the change of measure $\Gamma_t^k = \mathcal{E}_t(l(k)/\sigma \cdot W^0)$, setting $dW_t^k := dW_t^0 - l(k_t)/\sigma dt$.

The model and the agent's problem

The firm pays a cost $c(k)$ for technology ($c' \geq 0$) and also pays a fee $p(X_T)$ at maturity T if final emissions are too high (ex: $p(x) = \mathbf{1}_{[\Lambda,\infty)}(x)$ or $p(x) = (x - \Lambda)^+$). The expected utility of the agent with a given technology plan k and receiving incentives s can be written as

$$V(k) = V^{(s,p)}(k) = E^k \left[\int_0^T u(s_t - c(k_t))dt - p(X_T) \right]$$
$$= E \left[\int_0^T \Gamma_t^k u(s_t - c(k_t))dt - \Gamma_T^k p(X_T) \right] \tag{1.3}$$

Agent's value function: $v^{(s,p)} := \sup_{k \in \mathcal{A}} V^{(s,p)}(k)$.
$\mathcal{A}$: set of admissible effort policies.

Assumptions

Assumptions

The following assumptions hold:

1. l is $C_b^1(\mathbb{R}_+)$ and convex.
2. u is a utility function on $\mathbb{R}_+$, satisfying Inada conditions.
3. c is $\uparrow$ and convex on $\mathbb{R}_+$ with $c'(0) = 0$ and $c''(0) > 0$.
4. $p(X_T) \in L^{2+\alpha}$, for some $\alpha > 0$
5. $c(0) < m \leq s_t \leq M$ for some $0 < m < M$.

For the agent's problem, we fix a couple (s, p) (incentive structure).

Admissibility

Admissible efforts

An admissible effort policy $(k_t)_{0\leq t\leq T}$ is a positive $\mathcal{F}_t$-adapted stochastic process such that

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] < \infty$$

for some $\alpha > 0$. It is *strongly* admissible if $k_t \leq c^{-1}(s_t - \epsilon)$ a.s. $\forall t \in [0, T]$ for some $\epsilon > 0$.

- A strongly admissible effort policy is admissible.
- Since s is bounded, an admissible k is also bounded.
- If $u(0)$ is finite, then admissibility of k is equivalent to simply requiring $k_t \leq c^{-1}(s_t)$ a.s. $\forall t \in [0, T]$.

The agent's problem

State variable: Γ^k.
X: control independent.
Adjoint variables: (Y, Z) given by the (F)BSDE

$$\begin{cases} dX_t = X_t \sigma dW_t^0 \\ dY_t = -[Z_t l(k_t)/\sigma + u(s_t - c(k_t))]dt + Z_t dW_t^0 \\ X_0 = x, \, Y_T = -p(X_T) \end{cases} \tag{2.1}$$

Hamiltonian:

$$H(t, k_t, s_t, Z_t) = \Gamma_t^k [Z_t l(k_t)/\sigma + u(s_t - c(k_t))],$$

We can prove $Y_t = V_t^{(s,p)}(k)$, the conditional agent's expected utility.

The agent's problem: Optimality

Necessary conditions

Let k^* be an optimal strongly admissible control. Then there exist adapted processes (Y, Z) satisfying (2.1) with $k = k^*$ and the optimal control k^* satisfies

$$\begin{cases} \sigma u'(s_t - c(k_t^*))c'(k_t^*) = Z_t l'(k_t^*) \text{ on } \{k_t^* > 0\} \\ \sigma u'(s_t - c(k_t^*))c'(k_t^*) \geq Z_t l'(k_t^*) \text{ on } \{k_t^* = 0\} \end{cases} \quad (2.2)$$

Proof by the Stochastic Maximum Principle, using boundedness of s and strong admissibility to meet assumptions.

The agent's problem: Optimality

Necessary conditions/2

If the principal wants to induce a technology plan k, it is necessary to act on (s, p) in such a way that the volatility process in (2.1) satisfies

$$\begin{cases} \hat{Z}_t(s,p,k) = \sigma \frac{u'(s_t - c(k_t))c'(k_t)}{l'(k_t)} =: g(s_t, k_t) \text{ if } k_t > 0 \\ \hat{Z}_t(s,p,k) \geq 0 \text{ if } k_t = 0 \end{cases} \tag{2.3}$$

Necessary and sufficient conditions

A strongly admissible effort k is optimal for the agent given the incentive structure (s, p) if and only if the (unique) process Z defined as in (2.1) with (s, p, k) satisfies (2.3).

Interpretation

If (s, p, k) is fixed and Markovian, under some regularity $Y_t = \theta^{(s,p,k)}(t, X_t)$ where θ solves

$$\begin{cases} \theta_t + \frac{1}{2}\theta_{xx}x^2\sigma^2 + xl(k(t,x))\theta_x + u(s(t,x) - c(k(t,x))) = 0 \\ \theta(T,x) = -p(x) \end{cases} \tag{2.4}$$

Moreover $Z_t = \theta_x^{(s,p,k)}(t, X_t)\sigma X_t$.
Optimality conditions then impose

$$l'(k_t)\theta_x^{(s,p,k)}(t,x)\sigma x = u'(s_t - c(k_t))c'(k_t)$$

First term: marginal expected utility benefit from increasing effort.
Second term: marginal cost of effort.

Optimal effort: existence/uniqueness

Definition of $F(s, z)$

Given $0 < m < s \leq M$ and $z \in \mathbb{R}$, there exists a unique $k = F(s, z)$ satisfying

$$\begin{cases} z = g(s, k) \text{ if } k > 0 \\ z \geq 0 \text{ if } k = 0 \end{cases} \tag{2.5}$$

The function $F(s, \cdot)$ is nonincreasing, Lipschitz and continuously differentiable on $\mathbb{R} \setminus \{0\}$. Concave in z for common examples.

It "inverts" the constraints in (2.3).

The agent's problem: existence/uniqueness

If Z solves (2.1) and $k_t = F(s_t, Z_t)$ then k is an optimal response to incentives (s, p). Therefore (2.1) becomes

$$\begin{cases} dX_t = X_t \sigma dW_t^0 \\ dY_t = [-Z_t l(F(s_t, Z_t))/\sigma - u(s_t - c(F(s_t, Z_t)))]dt + Z_t dW_t^0 \\ X_0 = x, \, Y_T = -p(X_T) \end{cases} \tag{2.6}$$

Therefore the solution to this equation (when it exists) gives us the optimal effort by setting $k_t = F(s_t, Z_t)$.

Existence and uniqueness

There exists an admissible optimal effort k^* for the agent's problem. Uniqueness holds in the class of strictly admissible policies.

Existence/uniqueness: proof

It is enough to check that $f(s,z) := u(s - c(F(s,z))) + zl(F(s,z))/\sigma$ is uniformly Lipschitz continuous in z. We have

$$f_z(s,z) = l(F)/\sigma + F_z[zl'(F)/\sigma - u'(s - c(F))c'(F)].$$

where F stands for $F(s,z)$.
If $z \leq 0$ the term in brackets is zero by definition of F, while if $z > 0$ then $F_z = 0$, therefore $f_z(s,z) = l(F(s,z))/\sigma$, which is bounded by assumption.
By admissibility of s and p and standard existence theorems equation (2.6) has a unique solution (Y,Z) and therefore $k_t = F(s_t, Z_t)$ is an optimal effort.

Comparison results: utility

Comparison on expected utility

Assume that we have two admissible incentive policies (s, p) and $(\bar{s}, \bar{p})$ such that $\bar{s}_t \geq s_t$ a.s. for all t and $\bar{p}(x) \leq p(x)$ for all $x \geq 0$. Then $v^{(\bar{s},\bar{p})} \geq v^{(s,p)}$. Moreover, if $\bar{p}(x) < p(x)$ on a set of strictly positive Lebesgue measure, or if $\bar{s}_t > s_t$ on a set of strictly positive measure $dt \times dP$, then $v^{(\bar{s},\bar{p})} > v^{(s,p)}$.

Proof by comparison theorems for BSDEs.
The effects of greater incentives on the optimal effort $k_t = F(s_t, Z_t)$ are less clear to examine (hard to establish how Z will react to the change).

Comparison results: effort

Simple case: **constant incentives** s.
Key: study the Z**-part of the BSDE** (2.6) (with some regularity), since $k_t = F(s, Z_t)$.

- Write $Y_t = L(t, X_t)$, so that $Z_t = L_x(t, X_t)\sigma X_t$.
- Differentiate BSDE (2.6) wrt x to get

$$d\nabla Y_t = -l(F(s, Z_t))\nabla Z_t dt + \nabla Z_t dW_t^0$$

- Use $\nabla Y_t = L_x(t, X_t)\nabla X_t$, which implies $Z_t = \sigma X_t(\nabla X_t)^{-1}\nabla Y_t = x\nabla Y_t$
- Identify $N_t = \nabla Z_t$ and finally write

$$\begin{cases} dZ_t = -\frac{l(F(s,Z_t))}{\sigma} N_t dt + N_t dW_t^0 \\ Z_T = -\sigma X_T p'(X_T) \end{cases} \quad (3.1)$$

Comparison results: effort

- Now using $k_t = F(s, Z_t)$ apply Ito's lemma and identify terms to get

$$\begin{cases} -dk_t = \left[G(s, k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t \right] dt - \Theta_t dW_t^0 \\ k_T = F(s, -\sigma X_T p'(X_T)) \end{cases} \tag{3.2}$$

where $G(s, k) = \frac{1}{2} \frac{g_{kk}(s,k)}{g_k(s,k)}$.

Assumption: The function $x \mapsto xp'(x)$ is bounded

- $\Rightarrow Z_T$ is bounded $\Rightarrow Z$ is bounded by (3.1) and the comparison theorem $\Rightarrow k$ is strongly admissible $\Rightarrow G$ is bounded and therefore (3.2) has a unique solution, i.e. the optimal effort.

Comparison results: effort and risk aversion

Consider the power utility function $u(x) = u^\gamma/\gamma$.

Risk aversion and the optimal effort

In the power utility case, if

- $$p'(x)x \leq \frac{c'}{\sigma|l'|}\left(c^{-1}(s-1)\right) \qquad (3.3)$$
  for all $x \in \mathbb{R}_+$,
- $c^{(3)}(k) \leq 0$, $l^{(3)}(k) \leq 0$,

then the optimal effort is decreasing in γ, therefore increasing in the risk aversion coefficient $1-\gamma$.

Comparison results: impatience rate

Modified problem with impatience rate δ:

$$V(k) = E^k \left[\int_0^T e^{-\delta t} u(s_t - c(k_t)) dt - e^{-\delta T} p(X_T) \right].$$

Now the optimal effort solves

$$\begin{cases} -dk_t = \left[-\delta \frac{g(t,s,k_t)}{g_k(t,s,k_t)} + G(t,s,k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t \right] dt - \Theta_t dW_t^0 \\ k_T = F(s, -\sigma X_T p'(X_T)) \end{cases}$$

Impatience rate and optimal effort

The optimal effort is decreasing in the impatience rate δ.

Changes more relevant as the time to maturity increases.

Numerical scheme

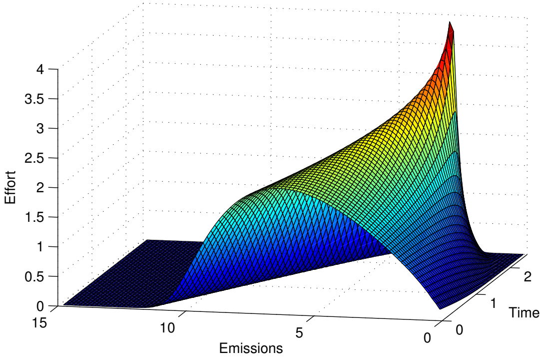
Idea: approximate Z and then recover k through $k_t = F(s, Z_t)$.
Write $Z_t = \phi(t, X_t)$ where ϕ solves (with some regularity)

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 x^2 \phi_{xx} + l(F(s,\phi))x\phi_x = 0 \\ \phi(T,x) = -\sigma x p'(x) \end{cases} \tag{4.1}$$

Standard numerical schemes available. Covergence of the order $\mathcal{O}(h + \Delta t)$.
Same rate for the optimal effort since F has bounded derivatives.

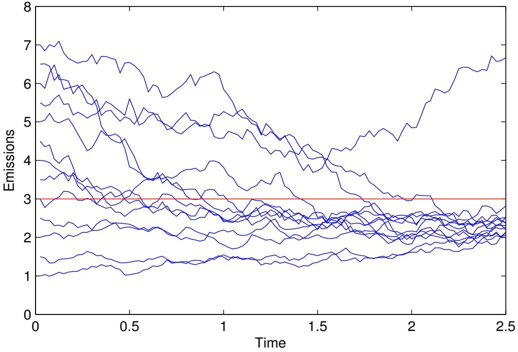
Example 1: Optimal effort structure

Take $l(k) = \frac{1-k}{1+k}$, $c(k) = k^2/2$ and $u(x) = 2\sqrt{x}$.
$p(x) = \mathbf{1}_{[3,\infty)}(x)$, $T = 2.5$. Constant incentives $s_t = 10$.



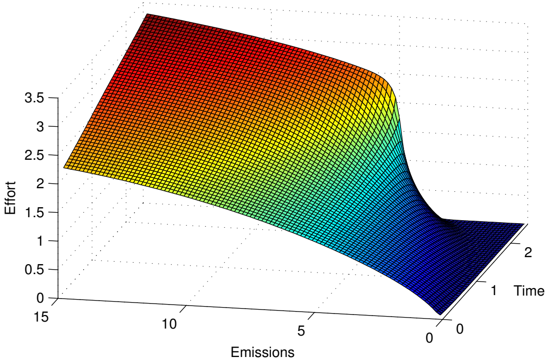
Example 2: Simulated trajectories

Simulation of tipical emissions paths following the agent's optimal effort of the previous example. The red line shows $\Lambda = 3$



Example 3: Optimal effort structure

$p(x) = (x-5)^+$. Constant incentives $s_t = 10$



The principal's problem

- Agent's problem: Given (s, p), find the best k.
- Principal's problem: Find the best (s, p), taking into account the agent's best response.

Principal's profit given (s, p, k):

$$E^k \left[p_1(p(X_T)) - p_2(X_T) - \int_0^T u_1(s_r)dr \right]$$

where

1. p_1 relates the final (dis)utility of the agent to the final utility of the principal.
2. p_2 captures the social costs related to the level of emissions.
3. u_1 is a utility function accounting for continuous-time payments to the agent.

The principal's problem

- Directly choosing (s, p) might be hard.
- Easier to optimize over (s, k), if the final fee p is taken so that the agent will actually choose k.
- Preliminary choice of agent's initial utility $R \in \mathbb{R}$ (otherwise it might go to $-\infty$).
- Recall Agent's utility

$$dY_t^A = [-Z_t^A l(k_t)/\sigma - u(s_t - c(k_t))]dt + Z_t^A dW_t^0$$

- Choosing initial (instead of final) condition now the SDE goes forward, and the terminal value Y_T^A will now be an output of the initial choice of (s, k), once Z^A is fixed.

The principal's problem

- We must also make sure that the resulting triplet $(s, -Y_T^A, k)$ is implementable. Set $Z_t^A = g(s_t, k_t)$ in the SDE wich becomes

$$Y_t^A = R - \int_0^t [u(s_r - c(k_r)) + g(k_r, s_r) l(k_r)/\sigma] dr + \int_0^t g(k_r, s_r) dW_r^0 \tag{5.1}$$

This way the triplet $(s, -Y_T^A, k)$ is the unique implementable policy for any strongly admissible (s, k).

The principal's problem

Example

Constant effort, incentives: $k_t = \bar{k}$, $s_t = \bar{s}$ ($\bar{k} < c^{-1}(\bar{s})$).
Final penalty to be proposed:

$$-Y_T^A = -R + \left[\bar{u} + \bar{g}\frac{\bar{l} + \sigma^2/2}{\sigma}\right] T + \frac{|\bar{g}|}{\sigma} \log \frac{X_T}{x}$$

where $\bar{g} = g(\bar{s}, \bar{k})$, $\bar{u} = u(\bar{s} - c(\bar{k}))$ and $\bar{l} = l(\bar{k})$.

- Even in this simple example, the final penalty is not of the form $p(X_T)$, since the initial value x of the emissions process appears in the formula.
- This kind of final fee penalizes the proportional increase in the emissions' level from the beginning of the period.

The principal's problem

Expected profit of the principal:

$$J(s,k) = E\left[\Gamma_T^k p_1(-Y_T^A) - \Gamma_T^k p_2(X_T) - \int_0^T \Gamma_r^k u_1(s_r)dr\right].$$

Principal's optimization problem: $v_P := \sup_{(s,k)} J(s,k)$.

- Solvable with SMP as before but with two state variables (Γ and Y^A)
- Sufficient conditions harder to derive (no convexity of Hamiltonian)
- We will consider the particular (and easier) case $p_1(x) = x$, i.e. when the final agent's disutility linked to the payment of the fee corresponds to a principal's utility of the same amount.

The principal's problem: the case $p_1(x) = x$

Principal's necessary conditions

An optimal control (s,k) satisfies

$$\begin{cases} u'(s_t - c(k_t)) - u_1'(s_t) = 0 & \text{on } \{m < s_t < M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \geq 0 & \text{on } \{s_t = M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \leq 0 & \text{on } \{s_t = m\} \\ l'(k_t) Z_t^P = 0 \quad \text{on } \{k_t > 0\} \\ l'(k_t) Z_t^P \leq 0 \quad \text{on } \{k_t = 0\} \end{cases} \tag{5.2}$$

where

$$\begin{cases} dY_t^P = [-l(k_t)/\sigma Z_t^P + u_1(s_t)]dt + Z_t^P dW_t^0 \\ Y_T^P = -Y_T^A - p_2(X_T) \end{cases} \tag{5.3}$$

The principal's problem: the case $p_1(x) = x$

Assumptions: Inversion

Suppose that

1. We are able to invert uniquely the first three conditions in (5.2) by constructing a continuous and a.e. differentiable function I such that $s = I(k)$ verifies them.
2. We can uniquely define a continuous and a.e. differentiable function $L(z)$ that solves (in k) the implicit equation $g(I(k), k) = z$ for all $z \leq 0$. We set $L(z) = 0$ if $z > 0$ (as we did with the function F in the preceding sections).

The principal's problem: the case $p_1(x) = x$

Suppose $k_t > 0$ at the optimum.

- Agent's optimality conditions:
  $Z_t^A = g(s_t, k_t) = g(I(k_t), k_t)$, hence $k_t = L(Z_t^A)$.
- Principal's optimality conditions: $Z_t^P = 0$, therefore

$$Y_0^P = -Y_T^A - p_2(X_T) - \int_0^T u_1(s_r)dr,$$

so that $-Y_T^A = p_2(X_T) + \int_0^T u_1(s_r)dr + c$ for some constant $c \in \mathbb{R}$.

Plugging this into the agent's BSDE we get

$$\begin{cases} dY_t^A = -[Z_t^A I(L(Z_t^A))/\sigma + u(I(L(Z_t^A)) - c(L(Z_t^A)))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p_2(X_T) - \int_0^T u_1(I(L(Z_t^A)))dt - c \end{cases} \tag{5.4}$$

The principal's problem: the case $p_1(x) = x$

Principal's necessary conditions/2

Suppose the strongly admissible contract (s^*, k^*) is optimal for the principal's problem with $p_1(x) = x$ and verifies $k_t^* > 0$ a.s. for all $t \in [0, T]$. Then there exists a solution (Y^A, Z^A) to (5.4) and $k_t^* = L(Z_t^A)$, $s_t^* = I(k_t^*) = I(L(Z_t^A))$.

Principal's sufficient conditions

Suppose that (5.4) admits a solution (Y^{A*}, Z^{A*}). Denote $k_t^* = L(Z_t^{A*})$ and $s_t^* = I(k_t^*)$. Then (s^*, k^*) is optimal for the principal's problem with $p_1(x) = x$.

The principal's problem: the case $p_1(x) = x$

Proof similar to the agent's case. Key idea: use the fact that (s_t^*, k_t^*) is the stationary point of $H^M(\cdot, \cdot, g(s_t^*, k_t^*))$, where

$$H^M(s, k, z) := -u_1(s) + u(s - c(k)) + zl(k)/\sigma.$$

In this way: no convexity requirements on H^M.

$$\begin{aligned}
&J(s,k) - J(s^*, k^*) \\
=&E^k\left[p_1(C_T^{(s,k)}) - p_2(X_T) - \int_0^T u_1(s_r)dr\right] - E^k\left[Y_0^{P*}\right] \\
=&E^k\left[\int_0^T [H^M(s_r, k_r, g(s_t^*, k_t^*)) - H^M(s_r^*, k_r^*, g(s_t^*, k_t^*))]dr\right] \le 0
\end{aligned}$$

An Example

Power utility function $u(x) = x^\gamma/\gamma$ and $u_1(x) = 2u(x)$.
Setting $K_m = c^{-1}(m(1 - 2^{\frac{1}{\gamma-1}}))$ now the function I takes the form

$$I(k) = \begin{cases} \frac{c(k)}{1-2^{\frac{1}{\gamma-1}}} & \text{if } K_m \leq k \leq K_M \\ m & \text{if } 0 \leq k \leq K_m \\ M & \text{if } k \geq K_M. \end{cases}$$

Function L also well defined for $1/2 \leq \gamma < 1$. Defining $\tilde{Y}_t^A = Y_t^A + \int_0^t 2u(I(L(Z_r^A)))dr$ then (5.4) becomes

$$\begin{cases} -d\tilde{Y}_t^A = \tilde{f}(Z_t^A)dt - Z_t^A dW_t^0 \\ \tilde{Y}_T^A = -p_2(X_T) - c \end{cases} \tag{6.1}$$

with $\tilde{f}(z) = zI(L(z))/\sigma + u(I(L(z)) - c(L(z))) - 2u(I(L(z)))$.

An Example

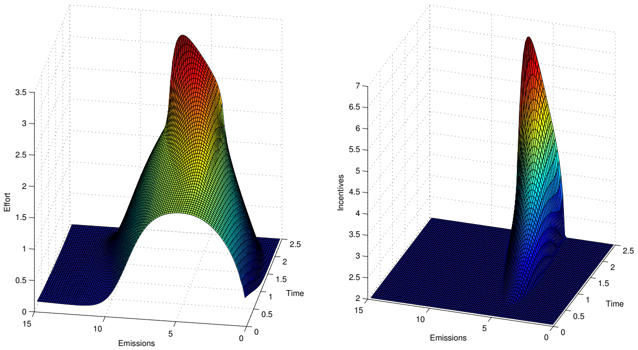
- Using the definitions of I and L we can show that $\tilde{f}'(z) = I(L(z))/\sigma$, which is bounded by definition. Hence if $p_2(X_T) \in L^2$ then (5.4) admits a unique solution and sufficient conditions work.
- As in the agent's case, under some regularity assumptions we can show that Z^A follows the BSDE

$$\begin{cases} dZ_t^A = -\frac{I(L(Z_t^A))}{\sigma} N_t dt + N_t dW_t^0 \\ Z_T^A = -\sigma X_T p_2'(X_T). \end{cases} \tag{6.2}$$

- Same numerical methods can be applied to recover the optimal quantities $k_t = L(Z_t^A)$ and $s_t = I(k_t)$.

An Example

Capped proportional penalty $p_2(x) = (x-4)^+ - (x-8)^+$, min/max incentives: $m = 2$, $M = 10$.



Thank you!