Existence and convergence of Glosten-Milgrom equilibra

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joint work with
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Market with insiders

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- How completely do prices reflect insider information?
- How large are insider profits?
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Market microstructure theory: [O'Hara]

Kyle model

Kyle (85) studied a market with single risky asset and obtained equilibrium between

- strategic informed investor [insider] whose trades move prices;
- other (uninformed) investors [noise trader] have random demand;
- a risk neutral market maker observe the aggregated demand and set the price as the conditional expectation of the risky asset.

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Kyle's model is quite influential:

- Kyle's λ^* measures market depth.
- Continuous time model has been studied by Back(92).
- Connection to filtration enlargement: Jacod, Jeulin, Yor, Protter ...
- Mathematical Finance: Pikovsky & Karatzas, Imkeller, Schweizer, Ankirchner, Amendinger, Monoyios, Campi, Cetin, Danilova ...

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Bid-ask spread exists because market maker wants to recoup the losses suffered in trading with informed trader.

Kyle meets Glosten and Milgrom

Back and Baruch, *information in securities markets: Kyle meets Glosten and Milgrom*, Econometrica, 72 (2004), pp. 433-465.

- Noise trades in Glosten-Milgrom is modeled by difference of two independent Poisson with intensity β and jump size δ.
- Noise trades in Kyle-Back is modeled by a Brownian motion.

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Back and Baruch showed

1. When $\delta \downarrow 0$ and $\beta \uparrow \infty$,

$$\text{Glosten-Milgrom equilibria} \quad \Longrightarrow \quad \text{Kyle-Back equilibrium.}$$

2. When δ is small,

$$\text{bid-ask spread} \quad \sim \quad 2\delta\lambda^*.$$

Our contributions

In Back-Baruch 04:

- HJB equations in Glosten-Milgrom are solved numerically.
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- hard-to-check conditions for convergence of equilibria.

Our contributions:

1. Optimal strategies for insider is explicitly constructed in Glosten-Milgrom.
2. A point process bridge is constructed.
3. Use weak convergence to remove technical assumptions for convergence.

Model

Interest rate 0.

One risky asset whose fundamental value is $\tilde{v}$.

$\tilde{v}$ has two states: high and low: 0 and 1 resp.

The fundamental value will be revealed at time 1.

Three types of market participants:

- Noisy/liquidity traders: total demand $Z = Z^B - Z^S$, Z^B/δ and Z^S/δ are independent Poisson with intensity β.
- Informed trader/insider: observes $\tilde{v}$ at time 0, net order $X = X^B - X^S$.
- Market maker: observe the aggregated order process $Y = X + Z$ and set the price at $\mathbb{E}[\tilde{v} \,|\, \mathcal{F}_t^Y]$.

Admissibility

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Insider's filtration $\mathcal{F}^I$ contains filtration generated by Z and $\tilde{v}$ + ?.

The insider's strategy (X^B, X^S) is admissible if

i) X^B and X^S, with $X_0^B = X_0^S = 0$, are $\mathcal{F}^I-$adapted increasing point processes with jump size δ;

ii) Z^B and X^B (resp. Z^S and X^S) never jump at the same time;

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Insider can either contributes or cancels noise trader's order.

$X^B = X^{B,B} + X^{B,S}$, where

- $X^{B,B}$: buy orders which compensate Z^B,
- $X^{B,S}$: buy orders which cancel some orders of Z^S.

Insider's problem

Insider maximizes profit associated to (X^B, X^S)

$$\int_0^1 X_{t-} dp(Y_t, t) + (\tilde{v} - p(Y_1, 1))X_1.$$

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This can be rewrite as

$$\begin{aligned}
&\int_0^1 (\tilde{v} - p(Y_t, t))dX_t^B - \int_0^1 (\tilde{v} - p(Y_t, t))dX_t^S \\
= \quad &\int_0^1 (\tilde{v} - p(Y_{t-} + \delta, t))dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-}, t))dX_t^{B,S} \\
&- \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t))dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-}, t))dX_t^{S,B}.
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&- \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t))d\theta_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-}, t))d\theta_t^{S,B} \\
&+ \text{martingales.}
\end{aligned}$$

Hence ask/bid prices can be defined

$$a(Y_{t-}, t) := p(Y_{t-} + \delta, t) \quad \text{and} \quad b(Y_{t-}, t) := p(Y_{t-} - \delta, t).$$

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HJB equation

Define value function $V(\tilde{v}, y, t)$,

$$
\begin{aligned}
& V_t + (V(y+\delta,t) - 2V(y,t) + V(y-\delta,t))\,\beta \\
& \quad + \sup_{\theta^{B,B}\geq 0} \left[V(y+\delta,t) - V(y,t) + (\tilde{v} - a(y,t))\,\delta\right]\theta^{B,B} \\
& \quad + \sup_{\theta^{B,S}\geq 0} \left[V(y,t) - V(y-\delta,t) + (\tilde{v} - p(y,t))\delta\right]\theta^{B,S} \\
& \quad + \cdots = 0.
\end{aligned}
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The system reduces to

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\begin{aligned}
V_t + (V(y+\delta,t) - 2V(y,t) + V(y-\delta,t))\,\beta &= 0, \\
V(y+\delta,t) - V(y,t) + (\tilde{v} - p(y+\delta,t))\delta &\leq 0, \qquad (1) \\
V(y-\delta,t) - V(y,t) - (\tilde{v} - p(y-\delta,t))\delta &\leq 0. \qquad (2)
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$$\begin{aligned} V_t + (V(y+\delta,t) - 2V(y,t) + V(y-\delta,t))\,\beta &= 0, \\ V(y+\delta,t) - V(y,t) + (\tilde{v} - p(y+\delta,t))\delta &\le 0, \quad (1) \\ V(y-\delta,t) - V(y,t) - (\tilde{v} - p(y-\delta,t))\delta &\le 0. \quad (2) \end{aligned}$$

- $\theta^{B,\cdot} > 0$ only when (1)$= 0$; $\theta^{S,\cdot} > 0$ only when (2)$= 0$;
- When one eqn. is identity, the other eqn. is strict inequality;
- $p \sim \partial_y V$ where V is a harmonic function.

Glosten-Milgrom equilibrium

A Glosten-Milgrom equilibrium is (p, X^B, X^S) such that

i) Given (X^B, X^S), $p(Y_t, t) = \mathbb{E}[\tilde{v} \mid \mathcal{F}_t^Y]$ for $t \in [0, 1]$;

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For $z \in \delta\mathbb{Z}$, let $P^z(y) = \mathbb{I}_{[y \geq z]}$ and

$$p^z(y,t) := \mathbb{P}[Z_1 \geq z \,|\, Z_t = y].$$

We expect $[\tilde{v} = 1] = [Y_1 \geq y_\delta]$ for some y_δ.

Theorem

(p^{y_δ}, X^B, X^S) is a Glosten-Milgrom equilibrium if

i) *$[Y_1 \geq y_\delta] = [\tilde{v} = 1]$ $\mathbb{P}$-a.s. for some $y_\delta \in \delta\mathbb{Z}$;*

ii) *$X^S \equiv 0$ on $[\tilde{v} = 1]$ ($X^B \equiv 0$ on $[\tilde{v} = 0]$);*

iii) *$Y = Z + X^B - X^S$ where Y^B/δ and Y^S/δ are independent, $\mathcal{F}^Y-$adapted Poisson processes with intensity β;*

Point process bridge

Assume $\delta = 1$.

We want to construct on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$

$$Y = Z^B - Z^S + X^B \mathbb{I}_I - X^S \mathbb{I}_{I^c},$$

$I \in \mathcal{F}_0$ and two point processes (X^B, X^S), such that

i) $I = [Y_1 \geq y_1]$ $\mathbb{P}-$a.s.;

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Similar to initial filtration enlargement where $\mathcal{G}_t = \mathcal{F}_t^Z \vee \sigma([Y_1 \geq y_1])$.

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However, standard theory gives

$$Y = Z^B - Z^S + \text{absolute cont. part}.$$

The absolute cont. part gives $\mathcal{G}$-intensities of Y^B and Y^S.

Explicit construction

Let us focus on I and before the first jump of Y.

Recall $p(y, t) = \mathbb{P}[Z_1 \geq y_1 \mid Z_t = y]$.

We want to construct, before the first jump of Y:

$$Y^B = Z^B + X^{B,B} \text{ with intensity } \beta \frac{p(1,t)}{p(0,t)} > \beta;$$

$$Y^S = Z^S - X^{B,S} \text{ with intensity } \beta \frac{p(-1,t)}{p(0,t)} < \beta.$$

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$X^{B,B}$ and $X^{B,S}$ are constructed to match the desired intensities.

- For $X^{B,B}$, construct ν_1 with $\mathbb{P}(\nu_1 > t) = \exp\left(\beta \int_0^t \left[1 - \frac{p(1,u)}{p(0,u)}\right] du\right)$;
- for $X^{B,S}$, accept jumps of Z^S at the rate $\beta \frac{p(-1,t)}{p(0,t)}$.

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Both can be achieved by introducing a sequence of iid uniform $[0,1]$ r.v.

Existence of Glosten-Milgrom equilibrium

Proposition

When $\mathbb{P}(I) = p(0,0)$*, then* Y *constructed above satisfies*

i) $[Y_1 \geq y_1] = I$ $\mathbb{P}$*-a.s.;*

ii) Y^B *and* Y^S *are independent Poisson with intensity* β *under the natural filtration of* Y*.*

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Theorem (Existence)

If there exists y_δ *such that*

$$\mathbb{P}(Z_1 \geq y_\delta) = \mathbb{P}(\tilde{v} = 1),$$

and $\mathcal{F}^I_t$ *includes filtration generated by* $\tilde{v}$*,* Z*, and a sequence of iid uniform. Then there exists a Glosten-Milgrom equilibrum.*

Kyle-Back model

In Kyle-Back model, demand of noise trader Z is a Brownian motion.

When $\tilde{v} = 0$ or 1, set $y_0 = \Phi^{-1}(1 - \mathbb{P}(\tilde{v} = 1))$ and pricing function

$$p^0(y,t) := \mathbb{P}^0_y[W_{1-t} \geq y_0].$$

Then the equilibrium demand satisfies

$$Y = W + \mathbb{I}_{[\tilde{v}=1]} \int_0^{\cdot} \partial_y \log p^0(Y_s, s)\, ds + \mathbb{I}_{[\tilde{v}=0]} \int_0^{\cdot} \partial_y \log(1 - p^0(Y_s, s))s, ds.$$

Y is Brownian motion conditional on $\mathbb{I}_{[W_1 \geq y_0]}$.

The insider's strategy is the additional drift in the enlarged filtration.

Convergence

When $\beta^\delta = (2\delta^2)^{-1}$,

$$Z^{B,\delta} - Z^{S,\delta} \xrightarrow{\mathscr{L}} W.$$

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Theorem (Convergence)

For any $\tilde{v}$ satisfying $\mathbb{P}(\tilde{v} = 1) \in (0,1)$, $\exists$ $(\tilde{v}^{\delta})_{\delta>0} \xrightarrow{\mathscr{L}} \tilde{v}$, s.t.

G-M equilibrum $(p^{\delta}, X^{B,\delta}, X^{S,\delta})$ exists whose fundamental value of risky asset is $\tilde{v}^{\delta}$.

When $\beta^{\delta} = (2\delta^2)^{-1}$, as $\delta \to 0$, G-M equilibria $\to$ K-B equilibrium:

i) *The bid-ask prices in G-M converge to the price in Kyle-Back.*

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \left(a^{\delta}(y,t) - p^{\delta}(y,t) \right) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \left(p^{\delta}(y,t) - b^{\delta}(y,t) \right) = \partial_y p^0(y,t).$$

ii) *When $\tilde{v} = 1$, $X^{B,\delta} \xrightarrow{\mathscr{L}} B^0$; when $\tilde{v} = 0$ $X^{S,\delta} \xrightarrow{\mathscr{L}} S^0$.*

Conclusion

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Thanks for your attention!