On the Hedging of Options On Exploding Exchange Rates

Johannes Ruf

Oxford-Man Institute of Quantitative Finance

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Peter Carr and Travis Fisher

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Goal

- Introduce pricing operator for contingent claims that satisfies put-call parity even if underlying X is a strict local martingale.
- E.g., relevant for the class of "Quadratic Normal Volatility" models:

$$\mathrm{d}X_t = (aX_t^2 + bX_t + c)\mathrm{d}W_t.$$

- First, we associate a probability measure to a given local martingale.

Digression: local martingales

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ is a *local martingale* if there exists a sequence of stopping times (τ_n) with $\lim_{n\to\infty} \tau_n = \infty$ such that X^{τ_n} is a martingale.

Probabilistic setup

- Time horizon $T > 0$.
- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, satisfying certain technical assumptions.
- In particular, $\mathbb{F}$ does not satisfy usual assumptions.
- Nonnegative $\mathbb{Q}$-local martingale X.
- Stopping times ($\inf \emptyset := T + 1$):

$$R_i := \inf\{t \in [0, T] : X_t \geq i\}, \qquad R = \lim_{i \uparrow \infty} R_i;$$

$$S_i := \inf\left\{t \in [0, T] : X_t \leq \frac{1}{i}\right\}, \qquad S = \lim_{i \uparrow \infty} S_i.$$

- Clearly, $\mathbb{Q}(R = T + 1) = 1$.
- Define the process $Y = \{Y_t\}_{t \in [0,T]}$ by $Y_t := 1/X_t \mathbf{1}_{\{R > t\}}$.

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- Define the process $Y = \{Y_t\}_{t \in [0,T]}$ by $Y_t := 1/X_t \mathbf{1}_{\{R > t\}}$.

Theorem: Change of measure, based on Föllmer (1972)

There exists a unique probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_{R-})$ such that

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[Z\mathbf{1}_{\{R>\tau\wedge T\}}\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[(Z\mathbf{1}_{\{S>\tau\wedge T\}})X_T^{\tau}\right]}{x_0};$$

$$\mathbb{E}^{\mathbb{Q}}\left[Z\mathbf{1}_{\{S>\tau\wedge T\}}\right] = x_0\mathbb{E}^{\widehat{\mathbb{Q}}}\left[(Z\mathbf{1}_{\{R>\tau\wedge T\}})Y_T^{\tau}\right]$$

for all stopping times τ and $\mathcal{F}_\tau$-measurable random variables $Z \in [0,\infty]$. Moreover, $\widehat{\mathbb{Q}}(R_i \wedge T < R) = 1$ for all $i \in \mathbb{N}$.

The process Y satisfies $\widehat{\mathbb{Q}}(\inf_{t\in[0,T]}\{Y_t\} \geq 0) = 1$; furthermore, $\widehat{\mathbb{Q}}(\inf_{t\in[0,T]}\{Y_t\} > 0) = 1$ if and only if X is a $\mathbb{Q}$-martingale. Moreover,

- Y is a $\widehat{\mathbb{Q}}$-supermartingale;
- Y is a local $\widehat{\mathbb{Q}}$-martingale if and only if $\mathbb{Q}(S > S_i \wedge T) = 1$ for all $i \in \mathbb{N}$;
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Example: singular measures

- Let X be a strict local martingale with dynamics

$$X_t = 1 + \int_0^{t \wedge S} \frac{1}{\sqrt{T-u}} \mathrm{d}W_u.$$

- We have $\mathbb{Q}(S < T) = 1$ and thus $\mathbb{Q}(X_T = 0) = 1$.
- For $Y := 1/X$ we get the $\widehat{\mathbb{Q}}$-dynamics

$$\mathrm{d}Y_t = -Y_t^2 \frac{1}{\sqrt{T-t}} \mathrm{d}W_t^{\widehat{\mathbb{Q}}}.$$

- We then obtain $Y_t > 0$ for all $t \in [0, T)$ and $Y_T = 0$; thus, $\widehat{\mathbb{Q}}(R = T) = 1$.
- The measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are singular with respect to each other since $\mathbb{Q}(R = T) = 0 < 1 = \widehat{\mathbb{Q}}(R = T)$.
- $\widehat{\mathbb{Q}}$ is absolutely continuous with respect to $\mathbb{Q}$ on $\mathcal{F}_t$ for all $t \in [0, T)$.
- X_t is a true $\mathbb{Q}$-martingale on $[0, t]$ for all $t \in [0, T)$, but a strict $\mathbb{Q}$-local martingale on $[0, T]$.

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Superreplicating price of a contingent claim

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
- Assume that the price of a tradable asset ("exchange rate") is given by a nonnegative process X.
- Given a contingent claim $D \geq 0$ (a function of the stock price path to the nonnegative reals).
- E.g., $D = 1$, $D = X_T$, $D = (X_T - K)^+$.
- If a bank sells such a contingent claim, the bank has to pay D Dollars to the buyer at time T.
- Question: Which price $\tilde{p}^{D,\$}$ should the bank charge for D?
- Possible answer (zero interest rate):

$$\tilde{p}^{D,\$} := \inf \Big\{ x \in \mathbb{R} : \exists \phi \text{ s.t. } x + \int_0^T \phi_s \mathrm{d}X_s \geq D \ \mathbb{P}\text{-a.s.}, \\ x + \int_0^t \phi_s \mathrm{d}X_s \geq 0 \ \mathbb{P}\text{-a.s.} \Big\}.$$

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Complete markets

- Assume there exists a unique probability measure $\mathbb{Q}^{\$}$, equivalent to $\mathbb{P}$, such that X is $\mathbb{Q}^{\$}$-local martingale.
- Then,

$$\tilde{p}^{D,\$} = \mathbb{E}^{\mathbb{Q}^{\$}}[D].$$

- Observe: For $D = X_T$, it is possible

$$\tilde{p}^{D,\$} < X_0.$$

- This is not an arbitrage opportunity since the trading strategy of selling asset and buying replicating portfolio is usually not admissible.

Risk management

- Assume, asset price has strict local martingale dynamics
  1. Replicating price of asset is smaller than asset price: $\mathbb{E}^{\mathbb{Q}^{\$}}[X_T] < X_0$.
  2. Put-call parity does not hold.
- Issues in risk management.
- Model output does not fit some market conventions (put-call parity).
- However, certain strict local martingale models (QNV models)
  - seem to fit very well to important FX market data (volatility smile);
  - are analytically tractable.
- Maybe the standard superreplicating price computed in such a model is not the right price one should charge?

Several suggestions in the literature

- Lewis (2000) proposes to add a correction term to the price of a call.
- Cox and Hobson (2005) suggest to consider collateral requirements when pricing contingent claims corresponding to a constraint on the class of admissible trading strategies.
- Madan and Yor (2006) propose to take the limit of a sequence of prices obtained from approximating the asset price by true martingales as the price for a contingent claim.

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A new price

- There exists a probability measure $\mathbb{Q}^{€}$, corresponding to the pricing measure of the European investor.
- However, if X is a strict local martingale, $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ are not equivalent.
- A joint superreplication price:

$$p^{D,\$} := \inf \Big\{ x \in \mathbb{R} : \exists \phi \text{ s.t. } x + \int_0^T \phi_s \mathrm{d}X_s \geq D \ \mathbb{Q}^{\$} \text{ and } \mathbb{Q}^{€}\text{-a.s.}, \\ x + \int_0^t \phi_s \mathrm{d}X_s \geq 0 \ \mathbb{Q}^{\$} \text{ and } \mathbb{Q}^{€}\text{-a.s.} \Big\}.$$

- Clearly, $p^{D,\$} \geq \tilde{p}^{D,\$}$.
- Another interpretation: superreplication under some physical measure $\mathbb{P}$, which is equivalent to $(\mathbb{Q}^{\$} + \mathbb{Q}^{€})/2$.

The economy

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{\$})$.
- For sake of simplicity, only two assets $(S^{\$,(0)}, S^{\$,(1)})$ to trade in: Dollars and Euros.
- Zero interest rates.
- Consider American investor: $S^{\$,(0)} \equiv 1$, $S^{\$,(1)} \equiv X$: $\mathbb{Q}^{\$}$-local martingales.
- Here, X: price of one Euro in Dollars.
- Assume that $\mathbb{Q}^{\$}(S_i < S) > 0$ for all $i \in \mathbb{N}$; that is, X does not jump to zero.
- There exists a probability measure $\mathbb{Q}^{€}$ corresponding to X as numéraire.
- $S^{€,(0)} := Y = 1/X\mathbf{1}_{\{R>\cdot\}}$ and $S^{€,(1)} \equiv 1$ are $\mathbb{Q}^{€}$-local martingales.
- Here, Y: price of one Dollar in Euros.

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Pricing measures and contingent claims

- We assume that $\mathbb{Q}^{\$}$ ($\mathbb{Q}^{€}$) is the unique local martingale measure for American (European) investor.
- A contingent claim is described by a pair $D = (D^{\$}, D^{€})$,
- satisfying $D^{€} = D^{\$}/X_T$ on the event $\{0 < X_T < \infty\}$.
- E.g.,
    - $D = (1, 1/X_T)$: one Dollar;
    - $D = (X_T, 1)$: one Euro;
    - $D = ((X_T - K)^+, (1 - K/X_T)^+)$: call on one Euro.
- Superreplicating price of D for American investor: $\mathbb{E}^{\mathbb{Q}^{\$}}[D^{\$}]$.
- Superreplicating price of D for European investor: $\mathbb{E}^{\mathbb{Q}^{€}}[D^{€}]$.

Wealth process and trading strategies

A *trading strategy* is an $\mathbb{R}^2$-valued, process $\eta \in L(S^\$) \cap L(S^€)$ such that

- its corresponding Dollar wealth process $(V_t^{\$,\eta})_{t\in[0,T]}$ and Euro wealth process $(V_t^{€,\eta})_{t\in[0,T]}$, defined by

$$V_t^{\$,\eta} := \eta_t^{(0)} + \eta_t^{(1)} X_t; \quad V_t^{€,\eta} := \eta_t^{(0)} Y_t + \eta_t^{(1)},$$

  stay nonnegative almost surely,
- the self-financing condition holds, that is,

$$\mathrm{d}V_t^{\$,\eta} = \eta_t^{(1)} \mathrm{d}X_t; \quad \mathrm{d}V_t^{€,\eta} = \eta_t^{(0)} \mathrm{d}Y_t.$$

Theorem: Joint superreplicating price

The minimal joint $\mathbb{Q}^{\$}$- and $\mathbb{Q}^{€}$-superreplicating price $p^{\$}(D)$ ($p^{€}(D)$) for a contingent claim $D = (D^{\$}, D^{€})$ is, expressed in Dollars (Euros),

$$p^{\$}(D) = \mathbb{E}^{\mathbb{Q}^{\$}}[D^{\$}] + x_0 \mathbb{E}^{\mathbb{Q}^{€}}\left[D^{€}\mathbf{1}_{\{1/X_T=0\}}\right],$$

$$p^{€}(D) = \mathbb{E}^{\mathbb{Q}^{€}}[D^{€}] + \frac{1}{x_0}\mathbb{E}^{\mathbb{Q}^{\$}}\left[D^{\$}\mathbf{1}_{\{X_T=0\}}\right] = \frac{p^{\$}(D)}{x_0}.$$

More precisely, there exists some trading strategy η for initial capital $p^{\$}(D)$ (expressed in Dollars) such that

$$\mathbb{Q}^{\$}\left(V_T^{\$,\eta} = D^{\$}\right) = 1 = \mathbb{Q}^{€}\left(V_T^{€,\eta} = D^{€}\right); \qquad (1)$$

and there exists no $\tilde{p} < p^{\$}(D)$ and no trading strategy $\tilde{\eta}$ for initial capital $\tilde{p}$ (expressed in Dollars) such that (1) holds with η replaced by $\tilde{\eta}$.

Some corollaries

- The minimal joint $\mathbb{Q}^{\$}$- and $\mathbb{Q}^{€}$-superreplicating price of $(X_T, 1)$ is x_0 (expressed in Dollars) or 1 (expressed in Euros).
- The put-call parity

$$p^{\$}(D_K^{C,\$}) + K = p^{\$}(D_K^{P,\$}) + x_0$$

holds, where $K \in \mathbb{R}$ denotes the strike of the call $D_K^{C,\$}$ and put $D_K^{P,\$}$.

- The pricing operators $p^{\$}$ and $p^{€}$ satisfy international put-call equivalence:

$$p^{\$}\left(D_K^{C,\$}\right) = x_0 K p^{€}\left(D_{\frac{1}{K}}^{P,€}\right);$$

$$p^{\$}\left(D_K^{P,\$}\right) = x_0 K p^{€}\left(D_{\frac{1}{K}}^{C,€}\right).$$

A different approach

- Upon now, we have started from a risk-neutral measure $\mathbb{Q}^{\$}$ of the American investor and then have constructed a risk-neutral measure $\mathbb{Q}^{€}$ of the European investor.
- Alternatively, we could start with specifying a physical probability measure $\mathbb{P}$.
- Let $\mathbb{P}$ denote any probability measure on $(\Omega, \mathcal{F}_T)$, possibly with explosions of X, that is, complete devaluations (hyperinflations) of Dollars and Euros, defined as

$$H^{\$} := \{X_T = \infty\} = \{R \le T\},$$
$$H^{€} := \{X_T = 0\} = \{S \le T\},$$

are both allowed to have positive probability under $\mathbb{P}$.

Digression: Hyperinflation

- *Hyperinflation*: complete devaluation of the corresponding domestic numéraire and an explosion of the exchange rate with respect to any other currency.
- Examples:
  - The price of one Dollar, measured in units of the respective domestic currency, went up by a factor of over 4500 in Austria from January 1919 to August 1922 and by a factor of over 10^{10} from January 1922 to December 1923 in Germany.
  - Hungary, August 1945 to July 1946. Prices soared by a factor of over 10^{27} in that 12-month period to which the month of July contributed a staggering raise of $4 * 10^{16}$ percent of prices.
  - Bolivia, August 1984 to August 1985: Price levels increased by $20,000$ percent.
  - Zimbabwe, July 2009: for instance, prices increased by an annualized inflation rate of over $2 * 10^8$ percent.

Replication under physical measure

- If indeed $H^{\$}$ and $H^{€}$ have positive probability there cannot exist a risk-neutral measure equivalent to $\mathbb{P}$ such that either X or $Y = 1/X$ follow local martingale dynamics.
- Define

$$\mathbb{P}^{\$}(\cdot) := \mathbb{P}(\cdot | H^{\$^C}) = \mathbb{P}(\cdot | R = T+1);$$
$$\mathbb{P}^{€}(\cdot) := \mathbb{P}(\cdot | H^{€^C}) = \mathbb{P}(\cdot | S = T+1).$$

- $\mathbb{P}$ is absolutely continuous with respect to their average $(\mathbb{P}^{\$}(\cdot) + \mathbb{P}^{€}(\cdot))/2$.
- Assume that $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ each allow for a unique equivalent local martingale measure, denoted again by $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$.

An extra condition

We shall utilize the assumption *No obvious hyperinflations (NOH)*:

(NOH) The probability measures $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ are equivalent on $\mathcal{F}_{(R_i \wedge S_j)-}$ for all $i, j \in \mathbb{N}$.

This corresponds to an environment in which, at no time, one knows that a certain hyperinflation will occur $\mathbb{P}$-almost surely; that is, hyperinflations occur as a surprise.

Proposition: Consistency conditions

Assume that (NOH) holds. Then the minimal replicating cost, in Dollars, for a contingent claim $D = (D^{\$}, D^{€})$ under $\mathbb{P}$ is exactly

$$p^{\$}(D) = \mathbb{E}^{\mathbb{Q}^{\$}}[D^{\$}] + x_0 \mathbb{E}^{\mathbb{Q}^{€}} \left[D^{€} \mathbf{1}_{\{1/X_T = 0\}} \right].$$

Moreover,

- $\mathbb{P}(H^{\$}) > 0$ if and only if X is a strict $\mathbb{Q}^{\$}$-local martingale;
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Conclusion

- We interpret the non-martingality of an asset price as a reflection of the possibility that the numéraire currency may devalue completely.
- We propose a new pricing operator for such models, under which put-call parity is restored.
- The pricing operator assigns prices to contingent claims according to the cost for replication strategies which succeed with probability one for both currencies as numéraire.

Thank you!

Digression: strict local martingales II

- Assume, asset price has strict local martingale dynamics.
- Consider the following trading strategy:
  - Sell stock for price x_0;
  - buy replicating portfolio of $D = X_T$ for price $\mathbb{E}^{\mathbb{Q}^\$}[X_T] < x_0$;
  - go on a nice vaccation with the profit $x_0 - \mathbb{E}^{\mathbb{Q}^\$}[X_T]$.
  - At time T, close your short position by liquidating the replicating portfolio.
- Arbitrage??
- Trading strategy is not admissible ("doubling strategy")!
- Thus, strict local martingales ("bubbles") are consistent with no-arbitrage condition.

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Digression: Change of numéraire

- Change of unit (currency) of asset prices
- Assume that X is a positive $\mathbb{Q}^{\$}$-martingale. Then, by Girsanov, there exists some probability measure $\mathbb{Q}^{€}$ s.t.

$$x_0 \mathbb{E}^{\mathbb{Q}^{€}}\left[\frac{Y}{X_T}\right] = \mathbb{E}^{\mathbb{Q}^{\$}}[Y]$$

  - Y: Dollar price of some asset
  - Y/X_T: Euro price ($\$/(\$/€) = €$)

  Furthermore, if (Y_t) is a $\mathbb{Q}^{\$}$-martingale, then (Y_t/X_t) is a $\mathbb{Q}^{€}$-martingale.
- What if X is a strict local martingale, possibly hitting zero?

Digression: put-call parity

- Assume zero interest rate and no dividends.
- A put with maturity T and strike K has payoff $C_T = (K - X_T)^+$
- A call with maturity T and strike K has payoff $P_T = (X_T - K)^+$
- Observe:

$$P_T + X_T = C_T + K$$

- Thus, one would expect put-call parity to hold.
- Model-free no-arbitrage relation?

Another digression: doubling strategies

- Infinite-time discrete-time and continuous-time models require some care for no-arbitrage.
- In discrete time: Roulette game at times $t_i = 1 - 1/n$
  1. Start with 1 Dollar, put it on red
  2. If you win, stop. Otherwise borrow 2 Dollars and put them on red.
  3. If you win, stop (you made 1 Dollar profit). Otherwise, borrow 4 Dollars and put them on red.
  4. ...
- To prevent such strategies, on has to impose some lower bounds on the wealth.