No-arbitrage in a numéraire independent modelling framework

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Outline of the talk

1. Motivation for numéraire independent modelling
2. Key ideas of numéraire independent modelling
3. Rigorous definitions and results


Motivation for numéraire independent modelling

Conceptual prelude

Concepts in Mathematical Finance

qualitative/ preference-independent	**quantitative/ preference-dependent**
meta-notion of arbitrage	utility maximisation
market completeness	quantile hedging
existence of bubbles ...	choice of numéraire ...

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- Mathematical finance **cannot** describe currency units or traded assets per se but only their relative prices in terms of another currency unit.
- Therefore, instead of speaking of "undiscounted" and "discounted" prices, one should better speak of prices "in an arbitrary currency unit" and "in a numéraire currency unit", respectively.

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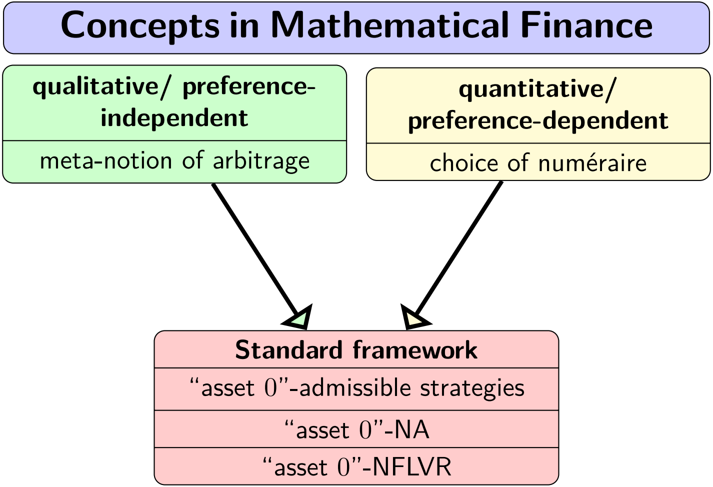
- For mathematical reasons, it is very convenient to choose a numéraire as the common currency unit. Which numéraire is chosen, is ex ante arbitrary, and as long as it remains so, no quantitative or preferential statement is made.
- **However**, as soon as a numéraire is chosen for a specific economical reason, e.g. the numéraire asset is interpreted as "riskless" or "credit line", an (implicit) qualitative or preferential statement is made.

The conceptual problem of the standard framework (1)



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- Take $\vartheta := (1, -1)$. Then ϑ is admissible and $\vartheta \bullet \widetilde{S}_1 = 2 - 1 = 1$.
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  $\implies$ $S^{X^{\$}}$ fails "\$-asset"-NA.
- **Questions:** Is the market per se **free of arbitrage or not**?
  Is the market free of arbitrage or not for a UK-citizen (no assets in $\pounds$!)?

Very short (and incomplete) survey of literature

- Delbaen and Schachermayer
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  - 1997: *The Banach space of workable contingent claims*

  These papers study in detail how "asset 0"-NFVLR relates to "asset 1"-NFLVR and how "asset 0"-maximal contingent claims relate to "asset 1"-maximal contingent claims.

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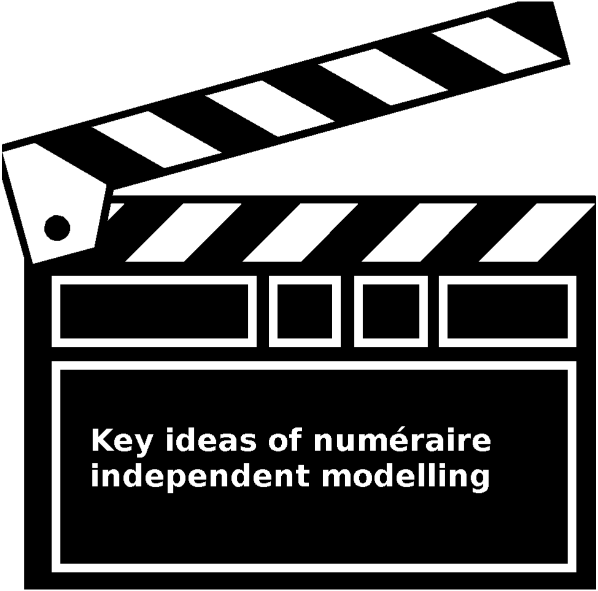
- **However**, very recently (and partly parallel to our research), there has been some work (Kardaras 2012, Takaoka 2012) which is mathematically but **not** conceptually in spirit of numéraire independent modelling.


Key ideas of numéraire independent modelling

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- **Fundamental paradigm of numéraire independent modelling:** Only consider properties of the market that hold for some representative $S \in \mathcal{S}$ if and only if they hold for all representatives $S \in \mathcal{S}$.

Self-financing, undefaultable and numéraire strategies

- **Trading** in a market $\mathcal{S}$ is described by **self-financing strategies**. For a self-financing strategy ϑ denote by $V_t(\vartheta)(S) := \vartheta_t \cdot S_t$ the **value process** of ϑ in units of S.

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- An undefaultable strategy η is called a **numéraire strategy** if $V(\eta)(S) > 0$ for some and hence every $S \in \mathcal{S}$. For each numéraire strategy, there exists a unique **numéraire representative** $S^\eta \in \mathcal{S}$ such that $V(\eta)(S^\eta) \equiv 1$.

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- **Numéraire independent FTAP:** A market $\mathcal{S}$ is free of "numéraire independent arbitrage" if and only if there exists a numéraire representative S^{η} and an equivalent σ-martingale measure for S^{η}.


Rigorous definitions and results

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- A nonempty set $\mathcal{S}$ of $\mathbb{R}^d$-valued semimartingales is called a **market** if

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and if some and hence every $S \in \mathcal{S}$ satisfies $\inf_{0 \leq t \leq T} \sum_{i=1}^{d} |S_t^i| > 0$.

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  and if some and hence every $S \in \mathcal{S}$ satisfies $\inf_{0 \leq t \leq T} \sum_{i=1}^{d} |S_t^i| > 0$.
- $S \in \mathcal{S}$ is called a **representative** of $\mathcal{S}$. It can be interpreted as a description of the market in some currency unit, which may be traded, e.g. €-, \$-bank account, or non-traded, e.g. Asian currency unit (ACU).

Self-financing and undefaultable strategies

Let $\mathcal{S}$ be a d-dimensional market and ϑ a d-dimensional predictable process

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- ϑ is called a **self-financing strategy** for $\mathcal{S}$ if for some and hence every $S \in \mathcal{S}$ and all stopping times $\tau \in \mathcal{T}_{[0,T]}$, $\vartheta \in L(S)$ and

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- A self-financing strategy ϑ is called **undefaultable**, notation $\vartheta \in \mathcal{U}(\mathcal{S})$, if $V(\vartheta)(S) \geq 0$ for some and hence every $S \in \mathcal{S}$. Undefaultable strategies are the appropriate substitute for admissible strategies in the standard framework.

Numéraire strategies, representatives and markets

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- If a numéraire strategy η exists, the market is called a **numéraire market**. **Note:** In nonnegative markets, $\eta := (1, \ldots, 1)$ is always a numéraire strategy.

Importance of numéraire strategies

- **Question:** If $S \in \mathcal{S}$ is a representative and $\vartheta \in L(S)$ how does one check in practice that ϑ is self-financing, i.e., satisfies $\vartheta_\tau \cdot S_\tau = \vartheta_0 \cdot S_0 + (\vartheta \bullet S)_\tau$ for all stopping times $\tau \in \mathcal{T}_{[0,T]}$?

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Let $\mathcal{S}$ be a numéraire market and η a numéraire strategy. Then for each $\zeta \in L(S^{(\eta)})$, there exists a self-financing strategy ϑ such that

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- **Conclusion:** Numéraire representatives are the "good" representatives in a market. Among them, none is "better" than the other, and the "standard" numéraire strategy $\eta := (1, 0, \dots, 0)$ is **not** special.

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- Either $\Pi(F) \equiv +\infty$ or $\Pi(F)$ is a contingent claim at time 0.

Maximal strategies

- **Idea:** We want to invest at time 0 into the market and trade until time τ by choosing a strategy $\vartheta \in \mathcal{U}(\mathcal{S})$. Independently of our personal preferences, ϑ can be considered a "reasonable investment" only if the following two conditions are satisfied:
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- **Useful fact:** A numéraire strategy η is strongly maximal if and only if its numéraire representative $S^{(\eta)}$ satisfies NFLVR.

Maximality of the zero strategy and no-arbitrage

- If the zero strategy is **not** strongly maximal, then there exists a stopping time $\tau \in \mathcal{T}_{[0,T]}$ and a non-zero contingent claim $F : \mathcal{S} \to \mathbf{L}^0_+(\mathscr{F}_\tau)$ such that

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Put differently, For each $S \in \mathcal{S}$, there exists a set $A \in \mathscr{F}_\tau$ with $\mathbb{P}[A] > 0$, such that for each $n \in \mathbb{N}$ and each $\epsilon > 0$, there exists $\vartheta \in \mathcal{U}(\mathcal{S})$ satisfying

$$V_0(\vartheta)(S) \leq \epsilon \quad \text{and} \quad V_T(\vartheta)(S) \geq n \ \text{ on } A.$$

This is closely related to the notion of cheap thrills, introduced by Loewenstein and Willard (2000).

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- If $\eta = (1, 0, \ldots, 0)$ is a numéraire strategy, then strong maximality of the zero strategy is equivalent to $S^{(\eta)}$ satisfying BK, introduced by Kabanov (1997), or no unbounded profit with bounded risk (NUPBR), introduced by Karatzas and Kardaras (2007).

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Theorem (dominating maximal strategies)

Let $\mathcal{S}$ be a numéraire market and suppose that 0 is strongly maximal. Then for each $\vartheta \in \mathcal{U}(\mathcal{S})$, there exists a dominating strategy $\vartheta^* \in \mathcal{U}(\mathcal{S})$ which is strongly maximal and satisfies

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- **Conclusion:** If the zero strategy is strongly maximal, then there exist "sufficiently many" strongly maximal strategies. More precisely, if $\vartheta \in \mathcal{U}(\mathcal{S})$ is **not** strongly maximal, one can always replace it by a "better" strategy ϑ^*.

Fundamental theorem of asset pricing

Theorem (standard FTAP, Delbaen and Schachermayer 1994, 1998)

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- **Conclusion:** If 0 is strongly maximal, then even if the original numéraire strategy η is **not** strongly maximal, we can replace it by a "good" strongly maximal numéraire strategy η^*, which is even better than η.

Example for numéraire independent FTAP

Let $W = (W_t)_{t\in[0,1]}$ be a Brownian motion. Define the stopping times

$$\tau_1 := \inf\left\{t > 0 : \int_0^t \frac{1}{1-s}\,\mathrm{d}W_s = -\frac{1}{4}\right\},\quad \tau_2 := \inf\left\{t > 0 : \int_0^t \frac{1}{1-s}\,\mathrm{d}W_s = -\frac{1}{2}\right\}.$$

Then $\tau_1 < \tau_2 < 1$. Define $S = (S_t^1, S_t^2)_{t\in[0,1]}$ (**no** riskless asset S^0) by

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- **Claim:** 0 is strongly maximal, $\eta^{(1)} := (1,0)$ and $\eta^{(2)} := (0,1)$ are **not** strongly maximal, and $\eta^* := (-1,2)\mathbb{1}_{[[0,\tau_1]]} + (2,0)\mathbb{1}_{((\tau_1,1]]}$ is a dominating (strongly maximal) numéraire strategy for $\eta^{(1)}$ and $\eta^{(2)}$.

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- **Conclusion:** Neither taking the first nor the second asset as numéraire is a good idea. Instead, work with $S^{(\eta^*)} = S$ because η^* is a "good" numéraire.

Thank you very much for your attention!


Questions?
Comments?