Price-Setting of Market Makers: A Filtering Problem with an Endogenous Filtration

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(Joint work with Christoph Kühn)

1. The model and the main result

2. Sketch of the proof

Market makers

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The spread is a compensation for certain risks the market maker face.

Information risk

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- Here, we follow the classic Glosten-Milgrom model (1985), who describe the prices as **expectations of a hidden true value** under the assumption of **risk-neutralilty** and **perfect competition** of market makers.

True value and customers

On a probability space $(\Omega, \mathcal{F}, P)$ we have

- the **true value process** of the asset $X = (X_t)_{t\geq 0}$, which is a time-homogeneous Markov process with finite state space $\{x_1, \ldots, x_n\}$, $n \geq 2$ where $x_{\min} = x_1 < \ldots < x_n = x_{\max}$ and transition kernel $q(i,j)$,
- the **customer arrival process** N, which is a Poisson-process with rate $\lambda > 0$, with jump times $\tau_1, \tau_2, \ldots$,
- and a sequence of iid random variables $(\epsilon_i)_{i\in\mathbb{N}}$. At τ_i the i-th customer arrives at the market and gets the **signal** $X_{\tau_i} + \epsilon_i$.

The processes and random variables are **mutually independent** from each other.

Buys and sells

- The **prices of the market maker** are modelled by an $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable mapping $S : \Omega \times [0,\infty) \to \mathbb{R}^2$. Write $S = (\overline{S}, \underline{S})$ to denote **ask and bid prices** with $\overline{S}_t(\omega) > \underline{S}_t(\omega)$ for all (ω, t).

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- The i-th customer buys if $X_{\tau_i} + \epsilon_i \geq \overline{S}_{\tau_i}$ and sells if $X_{\tau_i} + \epsilon_i \leq \underline{S}_{\tau_i}$. He does nothing if his valuation is within the spread.

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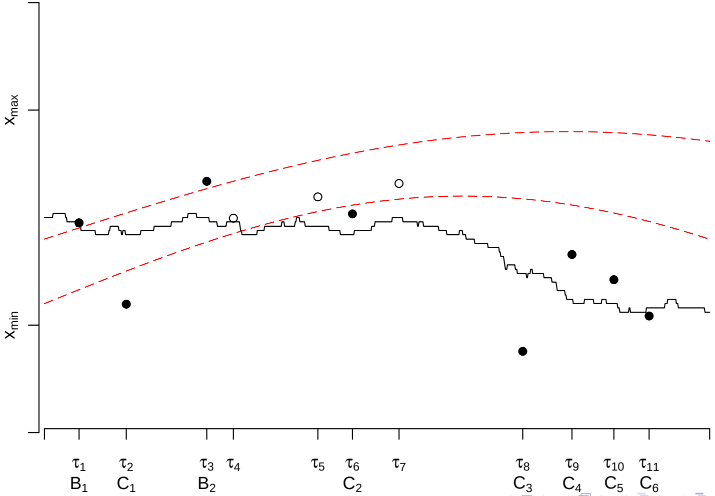
- The times of the *i*-**th buy** B_i and *i*-**th sell** C_i are given by

$$B_i := \inf\{\tau_j | \tau_j > B_{i-1}, X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j}\}, \qquad i \geq 1,$$

$$C_i := \inf\{\tau_j | \tau_j > C_{i-1}, X_{\tau_j} + \epsilon_j \leq \underline{S}_{\tau_j}\}, \qquad i \geq 1.$$

Buys and sells for given prices

black: true value, red: bid and ask prices, bullets: customers valuations



The filtration of the market maker

The **filtration of the market maker** is given by $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \geq 0}$, where

$$\mathcal{F}_t^S := \sigma(\{B_i \leq s\}, \{C_i \leq s\}, s \leq t, i \in \mathbb{N}).$$

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Definition

We say that S is an **admissible pricing strategy** if it is $\mathbb{F}^S$-predictable and $x_{\max} \geq \overline{S}_t(\omega) > \underline{S}_t(\omega) \geq x_{\min}$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.

Glosten-Milgrom prices

We now consider the price-setting under **perfect competition** and a **risk-neutral** market maker.

Definition

We say that an admissible pricing strategy S is a **Glosten-Milgrom pricing strategy (GMPS)** if for every bounded $\mathbb{F}^S$-stopping time τ

$$E\left[\sum_{B_i \leq \tau} (\overline{S}_{B_i} - X_{B_i})\right] = 0 \text{ and } E\left[\sum_{C_i \leq \tau} (\underline{S}_{C_i} - X_{C_i})\right] = 0.$$

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Lemma

S is a GMPS iff it is admissible and for all $i \in \mathbb{N}$

$$\overline{S}_{B_i} = E[X_{B_i} | \mathcal{F}^S_{B_i}] \qquad \text{and} \qquad \underline{S}_{C_i} = E[X_{C_i} | \mathcal{F}^S_{C_i}] \qquad P-a.s.$$

Main result

Theorem

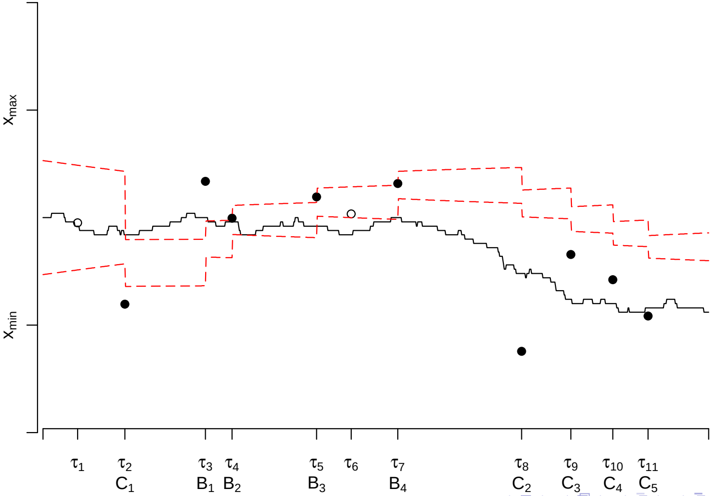
Let $C := x_{\max} - x_{\min}$ *and* $\Phi(y) := P[\epsilon_1 \geq y]$, $y \in \mathbb{R}$. *Assume that* Φ *is differentiable (i.e. the distribution of* ϵ_1 *has density* $-\Phi'$*) on* $[-C, C]$, $1 > \Phi(0) > 0$, *and*

$$-\Phi'(y) \leq \frac{K}{C} \min\{\Phi(y), 1 - \Phi(y)\}$$

for all $y \in [-C, C]$ *and a constant* $K < 1$. *Then,* **there exists a Glosten-Milgrom pricing strategy and it is unique** *up to a* $(P \otimes \lambda)$*-null set, where* λ *denotes the Lebesgue measure on* $\mathbb{R}_+$.

The Glosten-Milgrom pricing strategy

black: true value, red: bid and ask prices, bullets: customers valuations



Step 1: Existence and uniqueness at a fixed time

We consider the corresponding **static model** with distribution

$$P[\widetilde{X} = x_i] = \pi_i \qquad i = 1, \ldots, n$$

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The equilibrium condition for the **ask price** s is given by

$$E_\pi \left[(s - \widetilde{X}) 1_{\{\widetilde{X} + \epsilon \geq s\}} \right] = 0$$

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For $s \in [x_{\min}, x_{\max}]$ we define

$$g(s,\pi) := E_\pi\left[\widetilde{X} \middle| \widetilde{X} + \epsilon \geq s\right] := \frac{E_\pi\left[\widetilde{X} 1_{\{\widetilde{X}+\epsilon \geq s\}}\right]}{P_\pi\left[\widetilde{X} + \epsilon \geq s\right]}.$$

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The zero-profit condition translates to $g(s,\pi) = s$. The question of existence and uniqueness of a Glosten-Milgrom ask price is the same as the existence and uniqueness of a **fixed-point** of g.

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Lemma

Under the conditions on ϵ_1 from the main theorem there exists an unique static Glosten-Milgrom ask price s in $[x_{min}, x_{max}]$.

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Proof: Banach fixed-point theorem.

Lipschitz-contiuity in π

Lemma

Let all assumptions from above be fulfilled, then

$$|g(s,\pi) - g(\widetilde{s},\widetilde{\pi})| \leq K|s - \widetilde{s}| + L\sum_{i=1}^{n} |\pi_i - \widetilde{\pi}_i|$$

for $K < 1$ *from above and* $L < \infty$*, all* $s, \widetilde{s} \in [x_{\min}, x_{\max}]$ *and all distributions* $\pi, \widetilde{\pi}$.

Step 2: The solution as fixed-point

Lemma

For any $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $S = (\overline{S}, \underline{S})$, there exists a unique (up to indistinguishability) $\widetilde{\mathbb{F}}^S$-adapted càdlàg process π^S with

$$\pi^S_\tau = \left(P\left[X_\tau = x_i | \mathcal{F}^S_\tau\right]\right)_{i=1,\dots,n} \qquad P\text{-a.s.}$$

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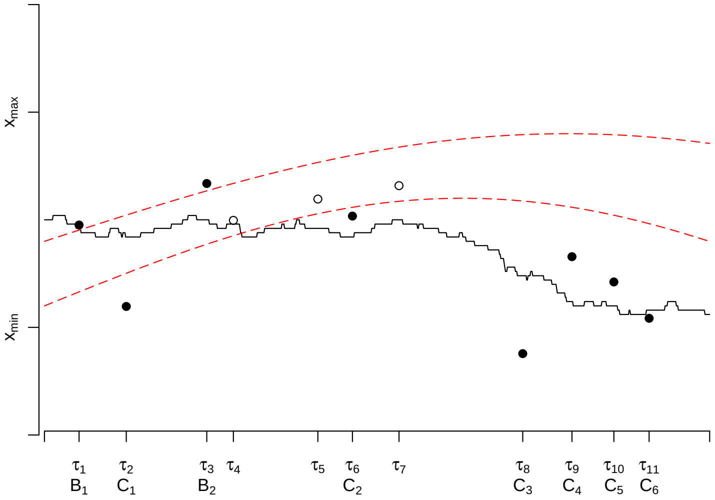
Definition

For admissible S we define $F(S) : \Omega \times [0,\infty) \to \mathbb{R}^2$ by

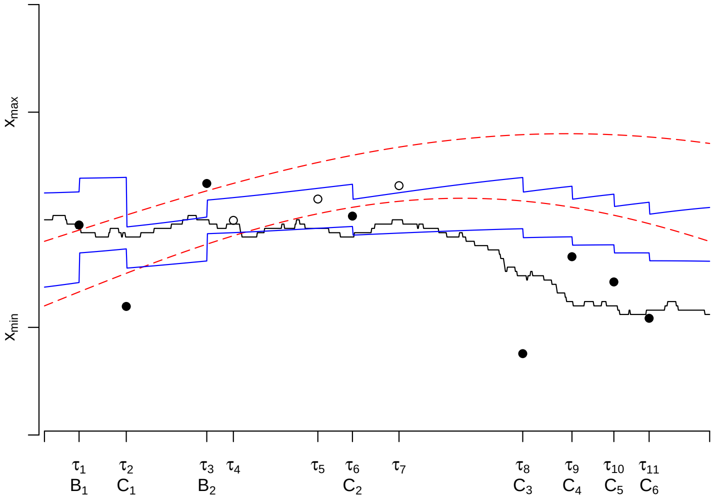
$$F(S)_t := \left(\overline{F(S)}_t, \underline{F(S)}_t\right) := \left(g\left(\overline{S}_t, \pi^S_{t-}\right), h\left(\underline{S}_t, \pi^S_{t-}\right)\right)$$

where g is defined as above and h is the analogon for bid prices.

Some pricing strategy S (in red)



Some pricing strategy S (in red) and $F(S)$ (in blue)



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Lemma

An admissible strategy S is a solution of the GMPS-problem iff S is a fixed point of F, i.e. $S = F(S)$ $(P \otimes \lambda)$-a.e. (where λ denotes the Lebesgue measure on $\mathbb{R}_+$).

Step 3: Uniqueness

Now we consider two admissible pricing strategies S and T. With some effort we obtain:

$$E\left[\int_0^t \sum_{i=1}^n |\pi_s^{S,i} - \pi_s^{T,i}| ds\right] \leq 6n\lambda MtE\left[\int_0^t \|S_s - T_s\| \, ds\right]$$

where $M := \max\{\Phi'(x)|x \in [-C, C]\}$.

Step 3: Uniqueness

Together with the Lipschitz-continuity of g this leads to

Lemma

There is a constant $K_1 < \infty$ such that

$$E\left[\int_0^t \|F(S)_s - F(T)_s\|\,ds\right] \leq (K + tK_1)E\left[\int_0^t \|S_s - T_s\|\,ds\right]$$

for all $t \geq 0$ and for $K < 1$ from the main result.

For small $t > 0$ (with $K + tK_1 < 1$) this yields a contraction and thus uniqueness in the $P \otimes \lambda$-sense up to t. Iteratively, uniqueness follows for $[0, \infty)$.

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The filter equation of the conditional probabilities is given by:

Step 4: Existence, Filter equation for $\pi_t^{S,i} = P[X_t = x_i | \mathcal{F}_t^S]$

$$
\begin{aligned}
d\pi_t^{S,i} =& \pi_{t-}^{S,i} \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\
&+ \pi_{t-}^{S,i} \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\
&- \Bigg(\lambda \pi_t^{S,i} \Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\
&- \sum_j \pi_t^{S,j} \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \Bigg) \\
&+ \sum_j \pi_t^{S,j} q(j,i) \Bigg) dt.
\end{aligned}
$$

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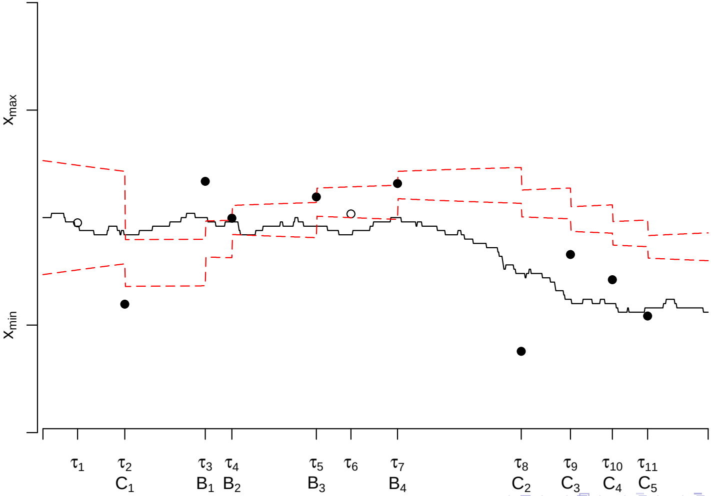

Thank you for your attention!