Optimal Investment and Consumption in a Mixed Liquid/Illiquid Market

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Introduction

- An important aspect of market liquidity : restriction on trading/observation times.
- Some previous works : market constituted of an illiquid asset that may only be traded/observed at discrete times.
  - Rogers (2001) : portfolio rebalanced at fixed time intervals of length h.
  - Rogers & Zane (2002), Matsumoto (2006), Pham & Tankov (2010) : trading times given by jump times of a Poisson process with constant intensity λ.

- Most of the litterature focuses on an agent investing exclusively in an illiquid asset. In practice, several correlated assets with different liquidity (for instance index fund and individual stocks).
- Longstaff (2005), Tebaldi & Schwartz (2006) : markets with one liquid and one illiquid asset, the latter may only be traded at the initial and terminal times.
- In our model, we take a less restrictive approach, and consider discrete (random) trading times for the illiquid asset. In this context, we study an investment/consumption problem over an infinite horizon.

Outline

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We consider a market consisting of a riskless asset assumed constant and two risky assets :

- a liquid asset that may be traded continuously, with price process L,
- an illiquid asset with price process I, that can only be traded at random times (τ_n) corresponding to the jump times of a Poisson process N with intensity λ. Between these random times it is only partially observed.

We assume that L and I follow Black-Scholes dynamics :

$$\begin{aligned} dL_t &= L_t(b_L dt + \sigma_L dW_t), \\ dI_t &= I_t(b_I dt + \sigma_I(\rho dW_t + \sqrt{1-\rho^2} dB_t), \end{aligned}$$

where W, B are independent BMs (independent of N), $\rho \in (-1, 1)$.
B is further decomposed as $B = \gamma B^{(1)} + \sqrt{1-\gamma^2} B^{(2)}$, for some observation parameter $\gamma \in [0, 1]$.

Observation filtration

We define the observation filtration for the agent :

$$\mathbb{G} := (\mathcal{G}_t)_{t \geqslant 0}; \quad \mathcal{G}_t = \sigma(\tau_n, I_{\tau_n}; \tau_n \leqslant t) \vee \mathcal{F}_t^W \vee \mathcal{F}_t^{B^{(1)}}.$$

i.e. at time t the agent has :

- full information on the past of the liquid asset up to time t,
- full information on the trading dates of the illiquid assets occurred before t and the values of the illiquid asset at such trading dates,
- partial information on the value of the illiquid asset up to time t (given by W and $B^{(1)}$).

Trading strategies

An investment strategy is then a triple (c, π, α), where

- $c = (c_t)_{t \geqslant 0}$, $\mathbb{G}$-predictable, nonnegative, represents the consumption rate;
- $\pi = (\pi_t)_{t \geqslant 0}$, $\mathbb{G}$-predictable, represents the amount of money invested in the liquid asset;
- $\alpha = (\alpha_k)$, is a discrete process where α_k is $\mathcal{G}_{\tau_k}$-measurable; represents the amount of money invested in the illiquid asset at times τ_k.

State processes

Wealth process R, dynamics :

$$
\begin{aligned}
R_0 &= r \geqslant 0, \\
R_t &= R_{\tau_k} + \int_{\tau_k}^{t} (-c_s ds + \pi_s (b_L ds + \sigma_L dW_s)) \\
&\quad + \alpha_k \left(\frac{I_t}{I_{\tau_k}} - 1 \right), \quad t \in [\tau_k, \tau_{k+1}).
\end{aligned}
$$

Denote by X (resp. A) the amount invested in liquid (resp. illiquid) wealth, i.e. for $t \in [\tau_k, \tau_{k+1})$:

$$
\begin{aligned}
X_t &= R_{\tau_k} - \alpha_k + \int_{\tau_k}^{t} (-c_s ds + \pi_s (b_L ds + \sigma_L dW_s)), \\
A_t &= \alpha_k \frac{I_t}{I_{\tau_k}}, \quad t \in [\tau_k, \tau_{k+1})
\end{aligned}
$$

<u>Remark</u> : A and R are not $\mathbb{G}$-adapted (unless $\gamma = 1$).

Admissible strategies

Initial wealth $r \geqslant 0$
$\mathcal{A}(r)$ set of strategies satisfying the no-bankruptcy constraint
$R_{\tau_k} \geqslant 0$ a.s., for each $k \geqslant 0$.
Under our assumptions, this is equivalent to the no-short-sale constraint

$$X_t \geqslant 0, A_t \geqslant 0 \quad \text{for every } t \geqslant 0,$$

or in terms of the strategy :

$$\begin{aligned} 0 &\leqslant \alpha_k \leqslant R_{\tau_k}, \quad \forall k \geqslant 0, \\ \int_{\tau_k}^{t} ((c_s - b_L \pi_s) ds - \sigma_L \pi_s dW_s) &\leqslant R_{\tau_k} - \alpha_k, \quad \forall t \in [\tau_k, \tau_{k+1}). \end{aligned}$$

Optimal investment/consumption

U increasing, concave function on $[0, \infty)$ with $U(0) = 0$.
$\beta > 0$ discount factor.

Value function :

$$V(r) = \sup_{(c,\pi,\alpha) \in \mathcal{A}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right].$$

The considered control problem is a mixed discrete/continuous control problem. As in Pham-Tankov (2008), by dynamic programming we reduce it to the study of a standard continuous (time-inhomogeneous) markovian control problem.

Outline

Dynamic Programming Principle

Proposition (DPP)

$$V(r) = \sup_{0 \leqslant a \leqslant r} \sup_{(\widetilde{c},\widetilde{\pi}) \in \mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(\widetilde{c}_s) ds + e^{-\beta \tau_1} V\left(R_{\tau_1}\right)\right].$$

Here for $x \geqslant 0$, $\mathcal{A}_0(x)$ is defined as the set of couples of stochastic processes $(\widetilde{c}_s, \widetilde{\pi}_s)_{s \geqslant 0}$ $\mathcal{F}_s^W \vee \mathcal{F}_s^{B^{(1)}}$-predictable such that

- $(\widetilde{c}_s)_{s \geqslant 0}$ is nonnegative,
- $(\widetilde{\pi}_s)_{s \geqslant 0}$ is locally square-integrable,
- $x + \int_0^T (-\widetilde{c}_s ds + \widetilde{\pi}_s (b_L ds + \sigma_L dW_s)) \geqslant 0, \quad \forall T \geqslant 0.$

Remark : the r.h.s. in the above is not quite a standard control problem (random horizon, partial observation).

We will rewrite the DPP further.
Recall $R_t = X_t + A_t$.
For $t \leqslant \tau_1$, we write $A_t = Y_t J_t$, where

$$Y_0 = \alpha_0, \qquad dY_s = Y_s \left(b_Y dt + \sigma_I(\rho dW_s + \gamma\sqrt{1-\rho^2} dB_s^{(1)}) \right),$$
$$J_0 = 1, \qquad dJ_t = J_t \left(b_J dt + \sigma_I \sqrt{1-\rho^2}\sqrt{1-\gamma^2} dB_t^{(2)} \right),$$

where b_Y, b_J are fixed and chosen s.t. $b_Y + b_J = b_I$.
(Actually we take $b_Y = \gamma^2 b_I + (1-\gamma^2)\frac{\rho b_L \sigma_I}{\sigma_L}$)
Then :

- Y is $\mathcal{F}^W \vee \mathcal{F}^{B^{(1)}}$-adapted,
- J is independent from $\mathcal{F}^W \vee \mathcal{F}^{B^{(1)}}$,
- for $t \leqslant \tau_1$, $R_t^{r,\widetilde{c},\widetilde{\pi},\alpha} = X_t^{r-\alpha_0,\widetilde{c},\widetilde{\pi}} + Y_t^{\alpha_0} J_t$.

Also define the operator G by :

$$G[\varphi](t,x,y) \;:=\; \mathbb{E}\left[\varphi(x + yJ_t)\right]$$

With these notations :

$$V(r) = \sup_{0 \leqslant a \leqslant r} \sup_{(\widetilde{c},\widetilde{\pi}) \in \mathcal{A}_0(r-a)} \mathbb{E} \int_0^\infty e^{-(\beta+\lambda)s} \left\{ U(\widetilde{c}_s) + \lambda G[V] \left(s, \widetilde{X}_s^{r-a,\widetilde{\pi},\widetilde{c}}, \widetilde{Y}_s^a \right) \right\} ds.$$

We now have a standard time-inhomogeneous (stochastic) control problem coupled with a one-dimensional extremum problem.

Dynamic value function

For $t, x, y \geqslant 0$, define

$$\widehat{V}(t,x,y) := \sup_{(\widetilde{c},\widetilde{\pi})\in\mathcal{A}_0(x)} \mathbb{E}\int_0^\infty e^{-(\beta+\lambda)s}\left(U(\widetilde{c}_s)+\lambda G[V]\left(t+s,\widetilde{X}_s^{x,\widetilde{\pi},\widetilde{c}},\widetilde{Y}_s^y\right)\right)ds.$$

V and $\widehat{V}$ are connected by

$$V(r) = \left[\mathcal{H}\widehat{V}\right](r) := \sup_{0\leqslant x\leqslant r}\widehat{V}(0,x,r-y).$$

Assumptions on U and β

- There exists $p \in (0,1)$, $K_1 > 0$ s.t.

$$U(c) \quad \leqslant \quad K_1 c^p, \quad x \geqslant 0.$$

- $\beta > k_M(p)$, where

$$k_M(p) := \sup_{u_L \in \mathbb{R}, u_I \in [0,1]} \left\{ p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_I^2\sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I) \right\}.$$

Proposition

V is concave, continuous and nondecreasing. Moreover

$$V(r) \leqslant Kr^p, \quad \textit{for some } K > 0.$$

For each t, $\widehat{V}(t,\cdot)$ is concave and nondecreasing with respect to both x, y. Furthermore, $\widehat{V}$ is continuous on $[0,+\infty) \times \mathbb{R}_+^2$, and for some $K > 0$,

$$\widehat{V}(t,x,y) \quad \leqslant \quad Ke^{k_J(p)t}(x+y)^p, \quad \forall (t,x,y) \in [0,+\infty) \times \mathbb{R}_+^2. \quad (1)$$

Moreover,

$$\widehat{V}(t,0,y) = \mathbb{E}\int_t^{\infty} e^{-(\beta+\lambda)(s-t)}\lambda G[V](s,0,\widetilde{Y}_s^{t,y})ds. \quad (2)$$

The HJB equation for our problem is written as

$$\begin{aligned} & -\widehat{V}_t + (\beta+\lambda)\widehat{V} - \lambda G[\mathcal{H}\widehat{V}](t,x,y) \\ & - \sup_{c \geqslant 0, \pi \in \mathbb{R}} H_{cv}(y, D_{(x,y)}\widehat{V}, D^2_{(x,y)}\widehat{V}; c, \pi) \quad = \quad 0, \end{aligned} \tag{3}$$

where the hamiltonian H_{cv} is defined by

$$\begin{aligned} & H_{cv}(y, D_{(x,y)}\widehat{V}, D^2_{(x,y)}\widehat{V}; c, \pi) \\ = \quad & U(c) + (\pi b_L - c)\widehat{V}_x + \frac{\rho b_L \sigma_I}{\sigma_L} y \widehat{V}_y \\ & + \frac{\sigma_L^2 \pi^2}{2}\widehat{V}_{xx} + \pi\rho\sigma_I\sigma_L y \widehat{V}_{xy} + \rho^2 \frac{\sigma_I^2}{2} y^2 \widehat{V}_{yy}. \end{aligned}$$

Viscosity characterization of the value function

Theorem

The value function $\widehat{V}$ *is a viscosity solution to the HJB equation* (3) *on* $[0,+\infty)\times(0,+\infty)\times\mathbb{R}_+$.
It is the unique solution satisfying the boundary condition (2) *and the growth condition* (1).

Outline

Motivation

We have a characterization of the value function, which allows to compute it numerically.
But to completely solve the control problem : we need existence/characterization of optimal strategies.
For that we want to apply verification theorems, and the value function $\widehat{V}$ needs to be a classical solution of the HJB equation.

In the general case, the HJB equation (3) is degenerate and it is not clear whether we have regularity. We restrict ourselves to the power utility case :

$$U(c) = c^p/p, \quad p \in (0,1).$$

For power utility, scaling property of the value function : $\widehat{V}(t, kx, ky) = k^p \widehat{V}(t, x, y)$.
Reduction to one space dimension : taking $z = \frac{x}{y}$,

$$\widehat{V}(t, x, y) \quad = \quad y^p \Phi(t, z).$$

Remark : we have assumed $y > 0$ (i.e. investment in the illiquid asset is nonzero). There is a simple condition insuring that $y = 0$ is not optimal :

Proposition

$\alpha_0^* > 0$ if and only if $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$.

This is the same condition as in the Merton model (without liquidity constraints).

HJB equation for Φ :

$$
\begin{aligned}
-\varphi_t + K_\lambda \varphi - K_3 z \varphi_z - \lambda f^\gamma(t,z)\mathcal{H}_0[\varphi] - \widetilde{U}(\varphi_z) \\
+\frac{1}{2}\frac{K_1^2}{K_2^2}\frac{\varphi_z^2}{\varphi_{zz}} - \frac{K_4^2}{2}z^2\varphi_{zz} &= 0, \quad (4)
\end{aligned}
$$

where

$$
\begin{cases}
K_\lambda = \beta + \lambda + \frac{\rho^2\sigma_I^2}{2}p(1-p) - p\rho\frac{b_L\sigma_I}{\sigma_L} - \gamma^2 p\left(b_I - \frac{\rho b_L \sigma_I}{\sigma_L}\right), \\
K_1 = b_L - \rho\sigma_I\sigma_L(1-p), \\
K_2 = \sigma_L, \\
K_3 = \gamma^2\left(-b_I + \frac{\rho b_L\sigma_I}{\sigma_L} + (1-\rho^2)(1-p)\sigma_I^2\right), \\
K_4 = -\sigma_I\gamma\sqrt{1-\rho^2}.
\end{cases}
$$

and

$$
f(t,z) := pG[U](t,z,1),
$$

$$
\mathcal{H}_0[\varphi] = \sup_{z>0} \frac{\varphi(0,z)}{(1+z)^p}.
$$

Regularity result

Theorem

Φ *is* $\mathcal{C}^{1,3}$ *on* $[0,+\infty) \times (0,+\infty)$.

Remark : in the proof we must distinguish the cases $\gamma \neq 0$ and $\gamma = 0$.

- For $\gamma \neq 0$: $K_4 \neq 0$ in (4), the PDE is strictly parabolic and we may apply standard regularity theorems (just need to prove that Φ_t, Φ_z, $(\Phi_z)^{-1}$, $(\Phi_{zz})^{-1}$ are locally bounded).
- For $\gamma = 0$: $K_4 = 0$ and the equation is degenerate. By convex duality, (4) is transformed into a quasi-linear parabolic PDE, for which we have regularity.

Auxiliary closed-loop equation

Define

$$\widetilde{C}^*(s,z) = \begin{cases} (U')^{-1}\left(\Phi_z(s,z)\right), & \text{if } z>0, \\ 0, & \text{if } z=0, \end{cases} \qquad \widetilde{\Theta}^*(s,z) = \begin{cases} -\frac{K_1\Phi_z(s,z)}{K_2\Phi_{zz}(s,z)}, & \text{if } z>0, \\ 0, & \text{if } z=0 \end{cases}$$

Proposition

Given $t \geqslant 0$, $z \geqslant 0$, there exists a unique solution $Z^{t,z,*} \geqslant 0$ to the closed loop state equation

$$\begin{aligned} dZ_s &= -\widetilde{C}^*(s,Z_s)ds + \widetilde{\Theta}^*(s,Z_s)\left(\widehat{K}_1 ds + K_2 dW_s\right) + Z_s(\widehat{K}_3 ds + K_4 dB_s^{(1)}) \\ Z_t &= z, \end{aligned} \tag{5}$$

where $\widehat{K}_1 = b_L - \rho\sigma_I\sigma_L$ and $\widehat{K}_3 = \gamma^2\left(-b_I + \frac{\rho b_L\sigma_I}{\sigma_L} + (1-\rho^2)\sigma_I^2\right)$.

Optimal strategy for the original problem

Proposition

Consider $t, x \geqslant 0$, $y > 0$. *Define* $z = \frac{x}{y}$, $Z^{t,z,*}$ *the solution to* (5), *and*

$$
\begin{aligned}
\widetilde{c}_s^{t,*} &= \widetilde{C}^*(t+s, Z_s^{t,z,*})/(Y_s^y)^p, \\
\widetilde{\pi}_s^{t,*} &= \widetilde{\Theta}^*(t+s, Z_s^{t,z,*})/(Y_s^y)^p + \frac{\rho\sigma_I}{\sigma_L} Z_s^{t,z,*}, \\
\widetilde{X}_s^{t,*} &= Y_s^y Z_s^{t,z,*}.
\end{aligned}
$$

Then $(\widetilde{c}^{t,*}, \widetilde{\pi}^{t,*}) \in \mathcal{A}_0(x)$, *with associated state process* $\widetilde{X}^{t,*}$, *and it is the unique optimal strategy, i.e. such that*

$$
\widehat{V}(t,x,y) = \mathbb{E}\int_0^\infty e^{-(\beta+\lambda)s}\left(U(\widetilde{c}_s^{t,*}) + \lambda G[V]\left(t+s, \widetilde{X}_s^{t,*}, \widetilde{Y}_s^y\right)\right) ds.
$$

Outline

Numerical computation : iterative procedure

- Start from $V^0 = 0$.
- Given V^n, take $\widehat{V}^n$ as the (unique) solution to the HJB where the non-local term is replaced by $\lambda G[V^n]$.
- Given $\widehat{V}^n$, take $V^{n+1} = \mathcal{H}\widehat{V}^n$.

Then as $n \to \infty$, $(V^n, \widehat{V}^n) \to (V, \widehat{V})$. Actually,

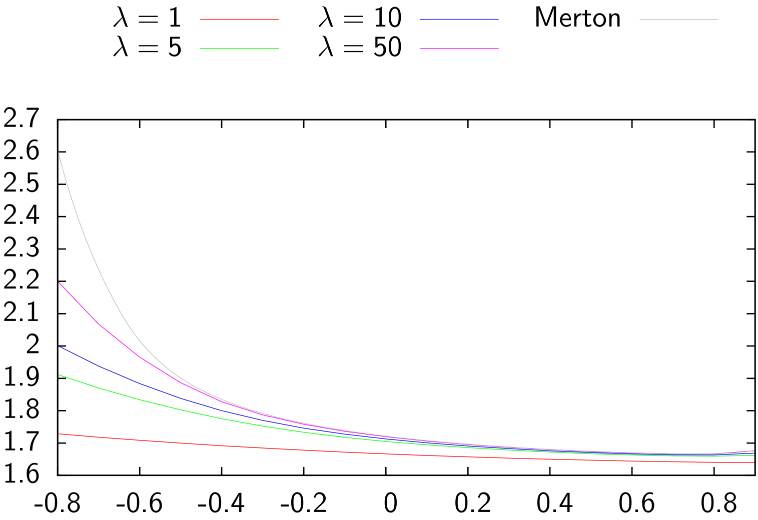
$$V^n(r) \quad = \quad \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\int_0^{\tau_n} e^{-\beta t} U(c_s)ds.$$

Chosen parameters : $U(c) = c^{1/2}$, and

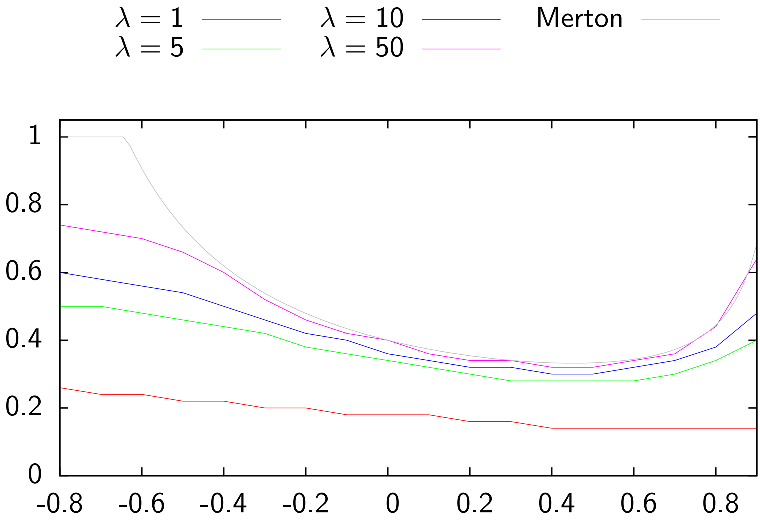
$$\beta = 0.2, \quad ; b_L = 0.15, \quad \sigma_L = 1, \quad b_I = 0.2, \quad \sigma_I = 1.$$

We make vary λ, ρ and γ.

Value function in function of ρ ($\gamma = 0$)



Optimal proportion in the illiquid asset in function of ρ ($\gamma = 0$)



Impact of the observation parameter γ

λ	1	5	10	50	Merton
$\gamma = 0$	1.66641	1.70493	1.71257	1.71945	1.72133
$\gamma = 1$	1.66995	1.71121	1.71656	1.72036	1.72133

Table: $V(1)$ for various γ, λ and fixed $\rho = 0$.

Cost of liquidity $e(r)$: $V(r + e(r)) = V_M(r)$.

λ	1	5	10	50
$\gamma = 0$	0.067	0.0193	0.0103	0.00218
$\gamma = 1$	0.062	0.0119	0.0056	0.00112

Table: $e(1)$ for various γ, λ and fixed $\rho = 0$.

We see that the relative impact of the observation constraint is higher for higher λ.