Efficient simulation and calibration of general HJM models by splitting schemes

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Why calibration? I

- client request for financial derivative
- task of sell-side trader:
  - quote a price
  - after the deal: hedge the institution's exposure due to the trade
- for both of those tasks: requires a model (obtaining price, perform hedge with respect to model parameters)

Why calibration? II

- in order to be able to hedge: choose sufficiently sophisticated model able to be close to true dynamics
- calibrate model parameters to the market
- hope: will remove arbitrage opportunities with respect to instruments used for calibration
- additionally: by calculating Greeks, able to perform hedges because the model not only predicts the risk-neutral price of the quoted derivative, but also the future evolution of the hedging instruments

The limits of calibration

- calibration is an expensive task: need to price many financial instruments often (inside optimisation routine)
- therefore: try to only calibrate to products that are easily and cheaply priced (ideally closed-form solutions!)
- problem: will reduce number of models we can choose
- additionally: might need different models to price different derivatives on the same underlying – potentially arbitrage opportunities in a single trader's portfolio!
- thus: want a single model (here: of interest rates)
  - matching statistical properties of the underlying
  - easy to calibrate
  - cheap to evaluate (for, e.g., exotic derivative)
- if statistical properties of underlying well-matched, might not need to recalibrate so often!

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Numerical discretisation of SDEs

- consider SDE on $\mathbb{R}^N$,

$$\mathrm{d}X(t,x) = \tilde{V}_0(X(t,x))\mathrm{d}t + \sum_{j=1}^{d} V_j(X(t,x))\mathrm{d}W_t^j, \quad X(0,x) = x$$

- usual discretisation: Euler scheme
- leads to issues for non-Lipschitz problems (Hutzenthaler, Jentzen, Kloeden 2011)
- strong rate 1/2 (important for multilevel Monte Carlo of Heinrich and Giles), weak rate 1
- our aim: method
    - of weak order 2
    - simple to implement

Splitting for ODEs

- consider ODE on $\mathbb{R}^N$,

$$\dot{X}(t,x) = f(X(t,x)) + g(X(t,x)), \quad X(0,x) = x$$

- if split equations

$$\begin{aligned} \dot{X}_1(t,x) &= f(X_1(t,x)), \quad X_1(0,x) = x, \\ \dot{X}_2(t,x) &= g(X_2(t,x)), \quad X_2(0,x) = x \end{aligned}$$

  easier to solve: approximate X using concatenation of the flows X_1, X_2
  - Lie-Trotter splitting: $X(t,x) = X_1(t, X_2(t,x)) + O(t^2)$
  - Strang splitting: $X(t,x) = X_1(t/2, X_2(t, X_1(t/2,x))) + O(t^3)$
- iteration yields global first (Lie-Trotter) or second (Strang) order

Ninomiya-Victoir splitting

- SDE in Stratonovich form,

$$\mathrm{d}X(t,x) = \sum_{j=0}^{d} V_j(X(t,x)) \circ \mathrm{d}W_t^j, \quad X(0,x) = x$$

- Markov semigroup $P_t f(x) := \mathbb{E}[f(X(t,x))]$
- split equations

$$\begin{aligned} \dot{X}_0(t,x) &= V_0(X_0(t,x)), \quad X_0(0,x) = x, \\ \mathrm{d}X_j(t,x) &= V_j(X_j(t,x)), \quad X_j(0,x) = x, \quad j = 1,\ldots,d \end{aligned}$$

- split Markov semigroups $P_t^j f(x) := \mathbb{E}[f(X_j(t,x))]$
- local discretisation error

$$P_t f(x) = \frac{1}{2} P_{t/2}^0 \left(P_t^1 \cdots P_t^d + P_t^d \cdots P_t^1 \right) P_{t/2}^0 f(x) + O(t^3),$$

at least for "sufficiently nice" f
- what happens in the $O(t^3)$?

Reduction to Kolmogorov equations

- by the forward Kolmogorov equation and Taylor expansion,

$$P_t f(x) = f(x) + t\mathcal{G}f(x) + \frac{1}{2}t^2\mathcal{G}^2 f(x) + \ldots$$

  with $\mathcal{G}$ the generator
- similarly for P_t^j using $\mathcal{G}^j$
- note $\mathcal{G} = \sum_{j=0}^{d} \mathcal{G}^j$ (at least formally)
- hence, for local error: need to bound $\mathcal{G}^{j_1}\mathcal{G}^{j_2}\mathcal{G}^{j_3}$ (locally, expand to third order)
- possible using
  - spaces of bounded and uniformly continuous functions, assuming bounded and smooth coefficients
  - spaces of functions with controlled growth, allowing the vector fields to be unbounded
- the second approach works for stochastic partial differential equations (non-locally compact state space)

The SPDE case I

- H Hilbert space, $\psi\colon H \to (0,\infty)$ such that $\{x \in H\colon \psi(x) \le R\}$ weakly compact for all $R > 0$
- $\mathrm{B}^{\psi}(H) := \{f\colon H \to \mathbb{R}\colon \|f\|_{\psi} < \infty\}$, where $\|f\|_{\psi} := \sup_{x\in H} \psi(x)^{-1}|f(x)|$
- $\mathcal{B}^{\psi}(H)$ space of all functions in $\mathrm{B}^{\psi}(H)$ that can be approximated by functions of the form $x \mapsto g(\langle y_1, x\rangle, \dots, \langle y_k, x\rangle)$ with $g\colon \mathbb{R}^k \to \mathbb{R}$ smooth and bounded with all derivatives bounded and $y_1, \dots, y_k \in H$
- Markov semigroups $(P_t)_{t\ge 0}$ on $\mathcal{B}^{\psi}(H)$ are strongly continuous
- how does the generator look like?

The SPDE case II

- to characterise generator $\mathcal{G}$: need spaces of differentiable functions $\mathcal{B}_k^{\psi}(H)$
- if X satisfies Da Prato-Zabczyk-SPDE

$$\mathrm{d}X(t,x) = AX(t,x)\mathrm{d}t + \tilde{V}_0(X(t,x))\mathrm{d}t + \sum_{j=1}^{d} V_j(X(t,x))\mathrm{d}B_t^j,$$

then for $\mathcal{B}_2^{\psi}(H)$,

$$\mathcal{G}f(x) = Df(x)(Ax) + Df(x)(\tilde{V}_0(x)) + \frac{1}{2}\sum_{j=1}^{d} D^2 f(x)(V_j(x), V_j(x))$$

- just as in finite-dimensional setting!
- similarly for Stratonovich SPDEs

The SPDE case III

- using these spaces of differentiable functions: can make $\mathcal{G} = \sum_{j=0}^{d} \mathcal{G}_j$ precise (holds true on twice differentiable functions)
- if we want to do Taylor expansion to order $O(t^3)$, require f 6 times differentiable in the above sense
- approximation $Q_{(\Delta t)} f := \frac{1}{2} P^0_{t/2} \left(P^1_t \cdots P^d_t + P^d_t \cdots P^1_t \right) P^0_{t/2} f$

Theorem (D., Teichmann 2010, 2011, D., Teichmann, Veluščček 2012)

If $f \in B_6^{\psi}(\tilde{H})$ on a sufficiently larger Hilbert space $\tilde{H}$, then

$$P_t f(x) - Q^N_{(t/N)} f(x) = O(N^{-2})$$

- proof works using arguments of E. Hansen, A. Ostermann for general operator splitting methods

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Interest rate modelling

- basic approaches to interest rate modelling:
  - short rate models: over-the-night interest rate $(R_t)_{t\geq 0}$, bond prices derived using $B(t,T) = \mathbb{E}[\exp(-\int_t^T R_x \mathrm{d}x)]$
  - term structure models: entire term structure of bond prices $(B(t,T))_{T\geq 0}$ for every $t \geq 0$ modelled directly
- advantages of short rate models:
  - low-dimensional
  - efficient pricing, calibration, hedging
- disadvantages:
  - low flexibility
  - not able to match complex initial interest rate curves consistently over time

HJM equation

- basic equation of term structure models:
  Heath-Jarrow-Morton equation
- stochastic partial differential equation
- in Musiela (time-to-maturity) parametrisation:

$$\mathrm{d}r_t(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} r_t(x) + \tilde{\alpha}_t(x)\right)\mathrm{d}t + \sum_{j=1}^{d} \sigma_t^j(x)\mathrm{d}W_t^j,$$

  x time to maturity, r_t forward rate, α drift, σ^j diffusions
- Da Prato-Zabczyk-type equation on suitable Hilbert space H (Filipovic: weighted Sobolev space on $[0, \infty)$)
- bond price:

$$B(t, T) = \exp\left(-\int_0^{T-t} r_t(\xi)\mathrm{d}\xi\right)$$

Why to use the HJM SPDE?

- advantages of the infinite-dimensional model:
  - very flexible
  - can match complex interest rate behaviour
  - can include jumps and stochastic volatility
- disadvantages:
  - complicated model
  - no closed-form solutions (for practically relevant choices of volatilities)
  - difficult to do pricing, calibration, hedging

Numerics for HJM

- our contribution: efficient numerical pricing and calibration
  - consider inherently infinite-dimensional model
  - development of effective algorithms
  - convergence analysis
  - implementation in C++
- other approaches:
  - T. Björk, A. Szepessy, R. Tempone, G. Zouraris 2002: Monte Carlo simulation and adaptivity
  - M. Krivko, M. Tretyakov 2011: efficient discretisation in space

A splitting approach to the HJM equation

- split up the equation into parts corresponding to the transport part, the HJM drift, and the diffusions
- solve those separately, one after the other (Strang, Ninomiya-Victoir, symmetrically weighted sequential splitting)

Theorem (D., Teichmann 2010, 2011)

Under appropriate smoothness assumptions, the scheme yields a weak approximation of order two: if $\hat{r}_t$ denotes the numerical approximation using N steps of the scheme,

$$\mathbb{E}[f(r_t)] - \mathbb{E}[f(\hat{r}_t)] = O(N^{-2})$$

for all sufficiently smooth functions $f : H \to \mathbb{R}$.

- consequence of the general result on splitting schemes

How does the splitting work?

- want to solve

$$\mathrm{d}r_t(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} r_t(x) + \alpha(r_t, x)\right) \mathrm{d}t + \sum_{j=1}^{d} \sigma^j(r_t, x) \circ \mathrm{d}W_t^j$$

- one after the other, solve

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} r_t^{01}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} r_t^{01}(x), & r_0^{01}(x) &= \rho(x), \\ \frac{\mathrm{d}}{\mathrm{d}t} r_t^{02}(x) &= \alpha(r_t^{02}, x), & r_0^{02}(x) &= \rho(x), \\ \mathrm{d}r_t^j(x) &= \sigma^j(r_t^j, x) \circ \mathrm{d}W_t^j, & r_0^j(x) &= \rho(x), \end{aligned}$$

- through correct concatenation: second weak order
- easy to implement
  - transport equation exactly solvable
  - ODEs on Hilbert space solvable using standard Runge-Kutta methods

what functions can we use?

- what weight function should we use?
- bond price: $B(t, T) = \exp(-\int_0^{T-t} r_t(\xi)\mathrm{d}\xi)$
- hence: choose $\psi(x) = \cosh(\beta \|x\|_H)$, then bond maturing at T in $\mathcal{B}^{\psi}(H)$ for β large enough
- works also for standard derivatives (caplets, swaps, swaptions etc.)
- derivatives with option component usually not differentiable
- nevertheless observe optimal rates of convergence numerically (in finite dimensions known, cf. smoothing properties, UFG condition, work of S. Kusuoka)

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A HJM model with stochastic volatility

- use model with independent randomness for stochastic, time-homogeneous, state-dependent, mean-reverting volatility,

$$\mathrm{d}r_t(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} r_t(x) + \alpha(r_t, x, v_t)\right)\mathrm{d}t + \sum_{j=1}^{d} \sigma^j(r_t, x, v_t)\mathrm{d}W_t^j,$$

$$\mathrm{d}v_t = -\alpha v_t \mathrm{d}t + \sum_{j=1}^{d} \gamma_j \mathrm{d}W_t^j$$

- Markovian dynamics
- flexible SABR-type volatility structure
- expected to allow good fits of given caplet data
- easy to calibrate to other products, e.g., swaptions

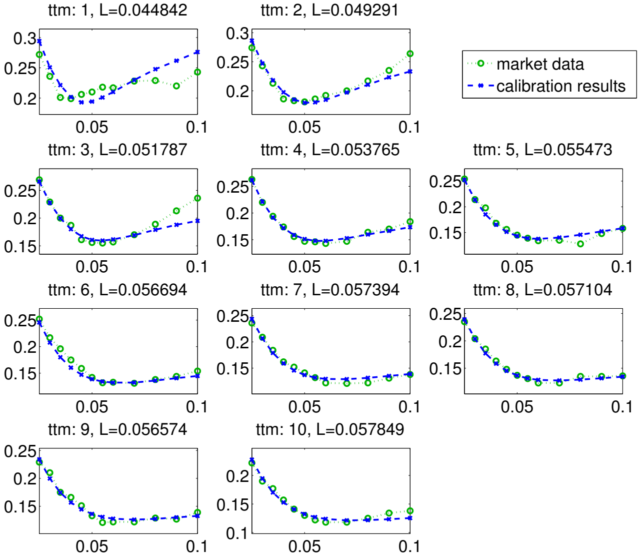
Calibration of the model

- parametrisation for the volatilities:

$$\sigma^j(r,x,v) = \tanh(c_j \exp(v) \int_0^{t_j} r(s)\mathrm{d}s)\lambda_j(x)$$

- fit market data of bond and caplet prices simultaneously
  - bond prices: reproduced exactly (input to the model!)
  - caplet prices: used for calibration of the parameters of the volatility vector fields
- 13 parameters used to match 120 prices (3 factors)
- calibration time: 14.5 minutes
  - 1743 evaluations, i.e., .5 seconds per calculation of 120 prices
  - 2048 quasi-Monte Carlo paths
  - 120 timesteps of the highly efficient second order splitting scheme

Calibration results (implied vol)



Remarks on the calibration

- good fit of medium and long maturities
- in order to fit short maturities better: jumps
- single model, i.e., leads to time-homogeneous dynamics
- should give more stable calibration, i.e., better hedges, no need to recalibrate all the time (only plug in the new forward rate curve and stochastic volatility)

Summary

- recap on calibration
- splitting schemes for SDEs and SPDEs
- advantages of HJM methodology over short rate models
- calibration of HJM model with stochastic, time-homogeneous, state-dependent volatility to real-world caplet data
- strategy applicable to other problems:
  - (multidimensional) SABR model
  - outside finance: stochastic Navier-Stokes equations
- code (partially) available online at `http://www.math.ethz.ch/~doersekp/was/`

Philipp Dörsek and Josef Teichmann.
Efficient simulation and calibration of general HJM models by splitting schemes.
ArXiv e-prints, December 2011,
http://arxiv.org/abs/1112.5330.