Portfolio Selection with Small Transaction Costs and Binding Portfolio Constraint

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October 8, 2012

Outline

The Portfolio Selection Problem (Continuous Time)

- Merton, starting in 1969:
  - Frictionless market consisting of one safe and one risky asset
  - Constant investment opportunities and CRRA for the investor
  - Maximize the expected utility of final wealth
  - **Solution**: risky weight $\pi_t \equiv \pi_*$

Merton's Problem with Frictions

Transaction Costs:

- Constantinides (1986):
  - No-trade region
  - First-order effect of transaction costs on asset's demand
  - Second-order effect of transaction costs on liquidity premium
- Shreve and Soner (1994)/ Janeček and Shreve (2004)/ Rogers (2004):
  - Width of no-trade region $\sim$ Transaction Cost$^{1/3}$
  - Liquidity Premium $\sim$ Transaction Cost$^{2/3}$

Binding Portfolio Constraint:

- Grossman and Vila (1992):
  - Reduced risky positions according to the portfolio constraint

Merton's Problem with both Frictions

- Dai, Jin and Liu (2011):
  - Maximizing the expected utility of final wealth
  - Solving the HJB equation by transformation to a double obstacle problem
  - Optimal trading boundaries obtained by numerical methods
- Our Goal: **more tractable results**

Tractable Model

- Dumas and Luciano (1991):
  - Constant investment opportunities
  - CRRA, i.e., $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, with $0 < \gamma \neq 1$
  - Consumption only at the terminal time
  - **Infinite horizon**
  - Look for strategies with optimal equivalent safe rates:

$$ESR^{\varphi} := \liminf_{T\to\infty} \frac{1}{T} \log U^{-1}(\mathbb{E}[U(\Phi_T^{\varphi})])$$

Φ_T: liquidation value of strategy φ at time T

The Model

The market:

$$\begin{cases} dS_t^0/S_t^0 = rdt, \quad S_0^0 = 1, \\ dS_t/S_t = (\mu + r)dt + \sigma dW_t, \quad S_0 = s_0 \in [0, \infty). \end{cases}$$

r: safe rate

μ: expected excess return of the risky asset

σ: volatility of the risky asset

$(W_t)_{t \geq 0}$: one-dimensional Brownian motion

Market Frictions

- **Transaction Costs**: bid-ask representation
  - Buy the risky asset using *ask-price* S
  - Sell the risky asset using *bid-price* $(1-\epsilon)S$ with $\epsilon \in (0,1)$
- **Portfolio Constraint**:
  - A trading strategy (ϕ^0, ϕ) satisfies the portfolio constraint, if

$$\pi_t := \frac{\phi_t S_t}{\phi_t^0 S_t^0 + \phi_t S_t} \leq \pi_{\max} =: \kappa \pi_*,$$

with the relative constraint $\kappa < 1$.

Long-run Optimality

Solve

$$\max_{(\psi^0,\psi)\in\mathcal{A}_{adm}} \left(\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[(\psi^0_T S^0_T + \psi^+_T (1-\epsilon) S_T - \psi^-_T S_T)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right)$$

An *admissible* trading strategy:

- self-financing
- solvent
- of finite variation
- satisfies portfolio constraint

Main Results: Optimal Policy

Theorem

Suppose $0 < \pi_{\max} \neq 1$:

- *Optimal to keep the risky weight (in terms of the ask price) within the buying and selling boundaries*

$$\pi_- = \pi_{\max}(1-\lambda), \quad \pi_+ = \pi_{\max},$$

where λ *is the root of a scalar equation and has the asymptotic expansion:*

$$\lambda = \left(\frac{(1-\pi_{\max})^2}{\gamma(\pi_* - \pi_{\max})}\right)^{1/2} \epsilon^{1/2} + \mathcal{O}(\epsilon),$$

as $\epsilon \downarrow 0$.

Rogers (2004): Width of the no-trade region

Transaction Costs Only:

- x : width of the no-trade region $\ni \pi_*$
- Loss due to transaction costs $\sim \epsilon/x$
- Loss due to suboptimal portfolio composition $\sim x^2$
- Minimize the total loss, i.e.,

$$\min_{x>0}(\epsilon/x + x^2) \rightsquigarrow x \sim \epsilon^{1/3}$$

Both Market Frictions:

- $\pi_* \notin$ no-trade region
- Loss due to suboptimal portfolio composition $\sim x$

$$\min_{x>0}(\epsilon/x + x) \rightsquigarrow x \sim \epsilon^{1/2}$$

Main Results: Equivalent Safe Rate

Theorem (continued)

Trading the risky asset with transaction costs and portfolio constraint is equivalent to:

- *leaving all wealth in a hypothetical safe asset with equivalent safe rate:*

$$\begin{aligned} ESR &= r + \frac{\mu^2}{2\gamma\sigma^2}(1-(1-\kappa(1-\lambda))^2) \\ &= r + \frac{\mu^2}{2\gamma\sigma^2}(2\kappa-\kappa^2) + \mathcal{O}(\epsilon^{1/2}). \end{aligned}$$

Main Results: Liquidity Premium

Theorem (continued)

- *trading a hypothetical asset, at no transaction costs and without portfolio constraint, with the same volatility* σ*, but with lower expected excess return* $\mu\sqrt{1-(1-\kappa(1-\lambda))^2}$*:*

$$\begin{aligned} LiPr &= \mu - \mu\sqrt{1-(1-\kappa(1-\lambda))^2} \\ &= \mu(1-\sqrt{2\kappa-\kappa^2}) + \mathcal{O}(\epsilon^{1/2}). \end{aligned}$$

Main Results: Share Turnover

Theorem (continued)

- *Share turnover (shares traded $d\|\varphi\|_t$ divided by shares held $|\varphi_t|$) has long term average:*

$$\begin{aligned} ShTu &:= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{d\|\varphi\|_t}{|\varphi_t|} \\ &= f(\mu,\sigma,\gamma,\kappa,\epsilon,\lambda(\epsilon)). \end{aligned}$$

Trading Volume: Share Turnover

Effects of the portfolio constraint:

- Reduced risky positions $\Rightarrow$ less trading volume
- Smaller no-trade region $\Rightarrow$ more trading volume
- $$ShTu = \gamma\sigma^2 \left(\frac{1}{\gamma}(\pi_* - \pi_{\max})(1 - \pi_{\max})^2 \right)^{1/2} \epsilon^{-1/2} + \mathcal{O}(1)$$
- For sufficiently small ϵ, $\epsilon^{-1/2} > \epsilon^{-1/3}$
  Portfolio constraint $\Rightarrow$ larger trading volume

Share Turnover vs. Bid-Ask Spread: No-Leverage Case

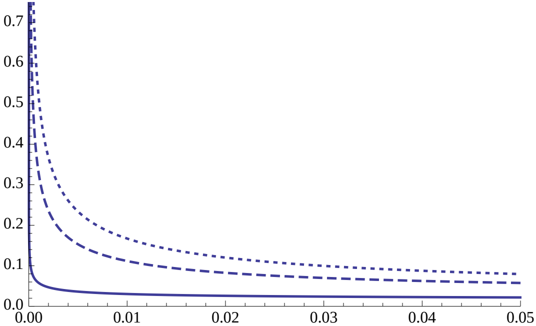

Figure: Solid: no constraint (unconstrained weight $\pi_* = 65\%$), Dashed: $\pi_{\max} = 50\%$, Dotted: $\pi_{\max} = 40\%$. Parameters are $\mu = 8\%$, $\sigma = 16\%$ and $\gamma = 5$.

Share Turnover vs. Bid-Ask Spread: Leverage Case

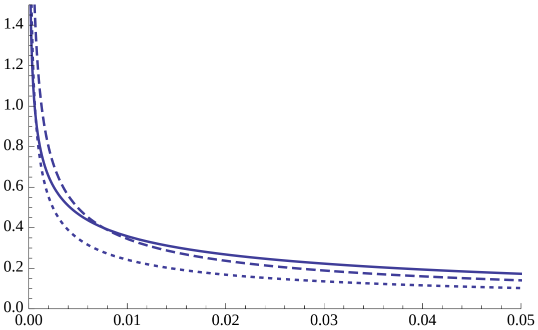

Figure: Solid: no constraint (unconstrained weight $\pi_* = 390\%$), Dashed: $\pi_{\mathsf{max}} = 225\%$, Dotted: $\pi_{\mathsf{max}} = 175\%$. Parameters are $\mu = 8\%$, $\sigma = 16\%$ and $\gamma = 0.8$.

Turnover, Spreads and Liquidity Premia

Transaction Costs Only (Gerhold, Guasoni, Muhle-Karbe and Schachermayer 2011):

$$LiPr \sim \frac{3}{4}\epsilon ShTu$$

Both Market Frictions:

$$LiPr^T := LiPr - \mu\left(1 - \sqrt{\frac{2\pi_{\max}}{\pi_*} - \left(\frac{\pi_{\max}}{\pi_*}\right)^2}\right)$$

$$\sim \left(\frac{\pi_{\max}}{2\pi_* - \pi_{\max}}\right)^{1/2} \epsilon ShTu$$

Application: Prime Broker Selection

Investing on Margin:

- Broker sets leverage constraint $\pi_{\max} > 1$ (i.e., margin requirement $= 1/\pi_{\max}$)
- Investor borrows from broker at lending rate r

No Transaction Costs: to achieve the same level of *esr*,

$$r = \frac{2\pi_{\max}\bar{\mu} - 2esr - \pi_{\max}^2 \gamma \sigma^2}{2(\pi_{\max} - 1)} \qquad (1)$$

$\bar{\mu}$ = total return of the risky asset

Application: Prime Broker Selection

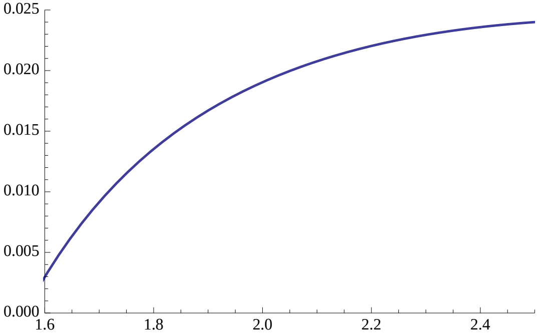

Figure: 10% esr indiffrence curve plotted against the interest rate r (vertical axis) and the constrained weight $\pi_{\max}$ (horizontal axis) for an unconstrained weight of 271%.

Application: Prime Broker Selection

- For pairs $(r, \pi_{\max})$ satisfying (1), the equivalent safe rate with transaction costs (at first order) is given by:

$$\mathrm{esr} - \left(\frac{\sigma^2}{2} \pi_{\max}^2 (\pi_{\max} - 1)(2\mathrm{esr} - 2\bar{\mu} - (\pi_{\max} - 2)\pi_{\max}\gamma\sigma^2) \right)^{1/2} \epsilon^{1/2}.$$

- Equivalent Safe Rate with Transaction Costs:
  - $\pi_{max} = 1 \Rightarrow$ no impact of transaction costs
  - $\pi_{max} = \pi_* \Rightarrow$ at first order no impact of transaction costs (non-trivial effect at order $\epsilon^{2/3}$)

ESR with Transaction Costs vs. π_{max}

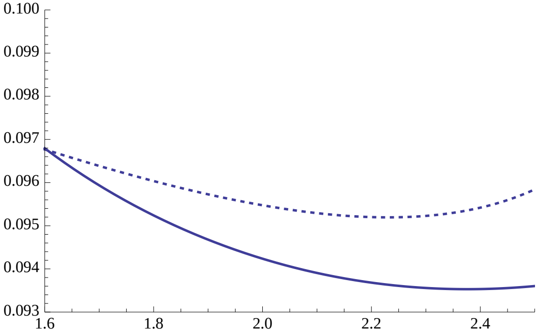

Figure: Solid: exact, Dashed: first-order approximation. r is chosen such that the equation (1) is satisfied for an unconstrained weight of 271%. Parameters are $\bar{\mu} = 8\%$, $\sigma = 16\%$, $\epsilon = 1\%$, $esr = 10\%$ and $\gamma = 0.8$.

Application: Illiquid Loans and Deposit Rates

Consider a bank,

- borrows from its depositors (safe rate r) and provides long-term loans, whose book values follow a geometric Brownian motion with drift $\bar{\mu}$ and volatility σ
- faces capital requirements (portfolio constraints) to limit excessive risk-taking
- aims to achieve the same performance/ESR for different capital requirements (tighter constraints offset by lower deposit rates)

Deposit Rates r

$\pi_{\max}$	ϵ	r	ESR
220%	0%	5.82%	10%
	0.1%	5.82%	9.84%
	1%	5.82%	9.54%
	10%	5.82%	8.86%
150%	0%	3.42%(−2.40%)	10%
	0.1%	3.60%(−2.22%)	9.84%
	1%	3.93%(−1.89%)	9.54%
	10%	4.68%(−1.14%)	8.86%

Table: Changes in deposit rates r due to a decrease of the upper bound of the risky weight from 220% to 150%, in order to retain the initial level of the equivalent safe rate. Model parameters are $\pi_* = 500\%$, $\bar{\mu} = 0.08$, $\sigma = 16\%$ and risk aversion is $\gamma = 0.1$.

Summary

- The portfolio constraint *reduces* the investment positions and typically *decreases* the width of the no-trade region.
- Transaction costs can have a *first-order* effect on the implied liquidity premium in the presence of a portfolio constraint.
- The welfare effect of small transaction costs is proportional to trading volume times the bid-ask spread.
- Prime Broker Selection: *In the presence of transaction costs the investor typically prefers tighter leverage constraints and lower lending rates.*
- Illiquid Loans and Deposit Rates: *Taking illiquidity into account highly leveraged positions are less attractive.*

Step 1: Heuristic Solution

- Ansatz: No-trade region $= [\pi_-, \pi_{max}]$
- Derive the HJB equation for the value function V
- Homotheticity and constant (long-run) growth rate of $V \Rightarrow$ free boundary problem
- Smooth pasting conditions (Dumas 1991) $\Rightarrow$ $\pi_- = \pi_{max}(1-\lambda)$, s.t. λ: solution of a scalar equation
- Implicit function theorem $\Rightarrow$ asymptotic expansions

Step 2: Construction of shadow market $(S^0, \tilde{S})$

Shadow Price Process $\tilde{S}$:

- Lies within the bid-ask spread $[(1-\epsilon)S, S]$ a.s.
- Existence of a long-run optimal strategy (φ^0, φ), i.e.,
  1. Finite variation strategy
  2. Self-financing strategy and solvent w.r.t. $\tilde{S}$
  3. Satisfies the original portfolio constraint
  4. Maximizes the equivalent safe rate w.r.t. $\tilde{S}$
  5. Entails buying only when $\tilde{S}_t = S_t$
  6. Entails selling only when $\tilde{S}_t = (1-\epsilon)S_t$

Step 2: Construction of $\tilde{S}$ in case of $\pi_{\max} < 1$

Idea: $\tilde{S}/S^0 =$ marginal rate of substitution of risky for safe assets

$$\frac{\tilde{S}_t}{S_t^0} = \frac{\partial_{\varphi_t} V(t, \varphi_t^0 S_t^0, \varphi_t S_t)}{\partial_{\varphi_t^0} V(t, \varphi_t^0 S_t^0, \varphi_t S_t)}$$

- $V =$ value function of the candidate strategy

Dynamics of $\tilde{S}$:

$$\frac{d\tilde{S}(Y_t)}{\tilde{S}(Y_t)} = (\tilde{\mu}(Y_t) + r)\, dt + \tilde{\sigma}(Y_t) dW_t + \left(1 - \frac{w'}{(1-w)w}\left(\log\left(\frac{u}{l}\right)\right)\right) dU_t$$

- $\tilde{\mu}$, $\tilde{\sigma}, w$: deterministic functions
- Y: drifted Brownian motion reflected at 0 and $\log(u/l)$
- U: local time process, only increasing at $\{Y_t = \log(u/l)\}$
- $[l, u]$: no-trade region reformulated for $\varphi S/\varphi^0 S^0$

Remark: Arbitrage Opportunity

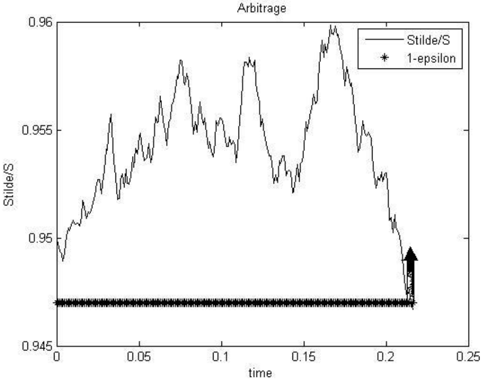

Figure: Solid: $\tilde{S}/S$, Star: 1-ϵ.

- $\tilde{S}_t = (1-\epsilon)S_t \Leftrightarrow Y_t = \log(u/l)$
- $\tilde{\pi}(\log(u/l)) = \pi_{\max}$

Step 3: Long-run Verification (Guasoni and Robertson 2011)

- Optimality of the candidate strategy in shadow market:
  - (super-) Martingale measure $\Rightarrow$ upper bound of the finite horizon ESR
  - Candidate strategy $\Rightarrow$ lower bound of the finite horizon ESR
  - Upper bound $=$ lower bound as $T \to \infty$
- Optimality of the candidate strategy in original market
  - Property of the shadow market