Dynamics of Corporate Security Prices in Firm Value Models with Incomplete Observation

Rüdiger Frey (based on joint work with Dan Lu)

WU Vienna

ruediger.frey@wu.ac.at
http://statmath.wu.ac.at/˜frey
www.math.uni-leipzig.de/˜frey

Berlin, October 2012

Structural credit risk models

In a typical structural or firm-value model default occurs if asset value V of a given firm falls below some threshold K, interpreted as liability. This leads to

$$\tau := \inf\{t \geq 0\colon V_t \leq K\}.$$

Typically V is modelled as a diffusion $\Rightarrow \tau$ is *predictable*.

Problems.

- The model gives unrealistically low short-term credit spreads.
- Asset value V is hard to observe precisely for investors on capital markets

Structural models with Incomplete information.

[Duffie and Lando, 2001]. V is geometric Brownian motion. Market observes at discrete time points t_n a signal $Z_n = \ln V_{t_n} + \varepsilon_n$ (termed *noisy accounting information*) and moreover the default event. We represent market information by filtration $\mathbb{F}^{\mathbb{M}}$

Results.

- Under incomplete information, τ is totally inaccessible and admits an *intensity* (an $\mathbb{F}^{\mathbb{M}}$-adapted process λ_t so that and $1_{\{\tau \le t\}} - \int_0^{t \wedge \tau} \lambda_s ds$ is an $\mathbb{F}^{\mathbb{M}}$ martingale).
- Characterization of intensity: $\lambda_t = \frac{1}{2}\sigma^2 K^2 \partial v \pi(t, K)$, $\pi(t, \cdot)$ the density of $\pi_{V_t | \mathcal{F}_t^{\mathbb{M}}}$.

Structural models with Incomplete information ctd.

[Frey and Schmidt, 2009]. Similar setup as in Duffie Lando.

- Dividends are introduced to study pricing of *equity* and debt.
- A systematic link between pricing corporate securities and (nonlinear) filtering: Prices are first computed under full information using Markov property, and then projected on market filtration $\mathbb{F}^{\mathbb{M}}$.
- The ensuing filtering problem is studied via a simple Markov-chain approximation.

Our Contributions [Frey and Lu, 2012]

- Noisy asset observation is modelled by a *continuous-time process* $Z_t = \int_0^t a(V_s)ds + W_t$
- A systematic analysis of the ensuing non-standard filtering problem using results of [Pardoux, 1978]
- We extend the Duffie-Lando characterization of default intensities to our setting.
- We derive the price dynamics for corporate securities in the market filtration $\mathbb{F}^{\mathbb{M}}$ (a prerequisite for derivative asset analysis under incomplete information).
- (Potential) applications: pricing and hedging of hybrid securities (convertibles or EDS) with credit risk, credit-valuation adjustments for counterparty risk.

Some literature

- Structural credit risk models: [Duffie and Lando, 2001], [Giesecke, 2004], [Jarrow and Protter, 2004], [Coculescu et al., 2008], [Frey and Schmidt, 2009], [Cetin, 2011],. . .
- Reduced-form models: [Collin-Dufresne et al., 2003], [Schönbucher, 2004], [Duffie et al., 2009] (empirical focus), [Frey and Runggaldier, 2010], [Frey and Schmidt, 2012]
- Surveys: [Frey and Runggaldier, 2008], [Frey and Schmidt, 2010], . . .

Overview

The filtering problem

Denote by $\mathcal{T} \subseteq \mathbb{R}^+$ a set of time points (usually $\mathcal{T} = \mathbb{R}^+$). Given an unobservable *signal or state process* $X = (X_t)_{t\in\mathcal{T}}$ and an *observation process* $Z = (Z_t)_{t\in\mathcal{T}}$, compute quantities of the form

$$E(\phi(X_t)|\mathcal{F}_t^Z), \phi \text{ bounded, measurable, } \mathcal{F}_t^Z = \sigma(Z_s : s \leq t), \quad (1)$$

in a *recursive* way. This means that $E(\phi(X_t)|\mathcal{F}_t^Z)$ is given in terms of 'sufficient statistic' $\pi_t, t \in \mathcal{T}$, i.e. $E(\phi(X_t)|\mathcal{F}_t^Z) = \delta(t, Z_t, \pi_t)$, where (π_t) can be updated using only new information:

$$\pi_{t+s} = \gamma(t, s, \pi_t, (Z_{t+u})_{0\leq u\leq s}) \quad (2)$$

(2) allows for quick updating of the filter. Typically, π_t is closely related to the conditional distribution of X_t given $\mathcal{F}_t^Z$.

Examples

Discrete time. Let $\mathcal{T} = (t_1, t_2, \dots)$, $\delta := t_i - t_{i-1}$ and X a discrete-time Markov chain. Observations $(\tilde{Z}_t)_{t \in \mathcal{T}}$ given by

$$\tilde{Z}_{t_i} = a(X_{t_i}) + \varepsilon_i, \varepsilon \; iid, \text{ independent of } X. \tag{3}$$

Often $\varepsilon_i \sim N(\sigma, \tilde{\sigma}^2)$, and $\tilde{\sigma}$ measures size of observation noise.

Continuous version. Here $X = (X_t)_{t \geq 0}$ Markov process in continuous time with generator $\mathcal{L}$ (think of an Ito diffusion), and observation process Z is given by

$$Z_t = \int_0^t a(X_s) ds + \sigma W_t, \tag{4}$$

W a BM independent of X. This can be viewed as continuous time analogue of (3), if we consider the *cumulative* observations

$$Z_{t_i} = \sum_{j \leq i} \delta \tilde{Z}_{t_j} = \sum_{j \leq i} a(X_{t_i}) \delta + \sum_{j \leq i} \delta \varepsilon_{t_i} \approx \int_0^{t_i} a(X_s) ds + \underbrace{\tilde{\sigma} \sqrt{\delta}}_{=:\sigma} W_t .$$

Reference probability approach

Here one uses a change of measure to study the filtering problem. Consider independent processes (X, Z) on $(\Omega, \mathcal{G}, Q^*)$ such that X is a Markov process with generator $\mathcal{L}$ and such that Z is a standard BM. Consider the density martingale $L_t = \frac{d\tilde{Q}}{dP}_{|\mathcal{F}_t}$ with

$$L_t = \exp\Big(\int_0^t a(X_s)^\top dZ_s - \frac{1}{2}\int_0^t |a(X_s)|^2\, ds \Big)\,. \tag{5}$$

Girsanov $\Rightarrow$ the pair (X, Z) has the right joint law under Q and we have

$$E^{\tilde{Q}}\big(f(X_t) \mid \mathcal{F}_t^Z\big) = \frac{E^{Q^*}\big(f(X_t)L_t \mid \mathcal{F}_t^Z\big)}{E^{Q^*}\big(L_t \mid \mathcal{F}_t^Z\big)} =: \frac{\Sigma_t f}{\Sigma_t 1}\,. \tag{6}$$

Zakai equation

One can show that for $f \in D(\mathcal{L})$ the measure-valued process Σ satisfies the following equation, known as *Zakai equation*

$$\Sigma_t f = \Sigma_0 f + \int_0^t \Sigma_s(\mathcal{L}f)ds + \int_0^t \Sigma_s(af)dZ_s.$$

- Zakai equation is linear "in Σ" and driven directly by observation Z.
- The equation is an infinite-dimensional measure-valued SDE
- Under regularity assumptions the Zakai equation gives rise to a SPDE for the Lebesgue density $u(t,\cdot)$ of Σ_t, driven by the adjoint of the adjoint of $\mathcal{L}$:

$$du(t) = \mathcal{L}^* u(t)dt + a^\top u(t)dZ_t. \tag{7}$$

Finite-dimensional approximation of Zakai equation

- Markov-chain approximation. If X is (approximated by) a finite state Markov chain with K states and transition intensities q_{ij} one gets the K-dimensional SDE system

$$d\Sigma_t^i = \sum_{k=1}^{K} q_{ki}\Sigma_t^k dt + \Sigma_t^i a(x_i)\, dZ_t,\ 1 \leq i \leq K.$$

- Galerkin approximation. This is based on the SPDE (7). One considers an approximation $\tilde{u}^{(m)}(t) = \sum_{i=1}^{m} \psi_i(t) e_i$ for smooth basis functions $e_1, \ldots e_m$ and one derives an SDE system for the m dimensional process $\Psi^{(m)}(t) = (\psi_1(t), \ldots, \psi_m(t))'$.
- Simulation methods such as particle filtering

References

- [Davis and Marcus, 1981] Good introduction
- [Bain and Crisan, 2009] Modern technical account, including numerics
- [Brémaud, 1981] Filtering with point-process observation
- [Budhiraja et al., 2007] Numerical methods

The Model

- We work on $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \geqslant 0}, Q)$; all processes are $\mathbb{G}$-adapted; Q is the martingale measure used for pricing
- Consider a company with asset value process $V = (V_t)_{t \geqslant 0}$ and default time $\tau = \inf\{t \geqslant 0 \colon V_t \leqslant K\}$.
- Company pays dividend d_n at date T_n, $n \geq 0$. T_n is nth jump time of a Poisson process with intensity λ^D and $d_n = \delta_n V_{T_n}$ for a iid sequence $(\delta_n)_{n=1,2,\ldots}$, independent of V, with mean $\bar{\delta}$.
- $D_t = \sum_{\{n \colon T_n \leqslant t\}} d_n$ is the cumulative dividend process and $\varphi(y, V_{T_n})$ denotes conditional density of d_n given V.
- Under Q, V is geometric Brownian motion, $dV_t = (r - \lambda^D \bar{\delta}) V_t dt + \sigma V_t dB_t$. Moreover V_0 has Lebesgue density $\pi_0(v)$ with $\pi_0(K) = 0$.

Market Information

The market uses the following pieces of information to price securities

- *Default information.* Market observes default state $N_t = 1_{\{\tau \leqslant t\}}$ of the firm. We denote the default history by $\mathbb{F}^N = (\mathcal{F}_t^N)_{t \geq 0}$.
- *Dividend information.* Market observes D_t with associated filtration $\mathbb{F}^D = (\mathcal{F}_t^D)_{t \geq 0}$.
- *Noisy asset observation.* Market observes a process Z with $Z_t = \int_0^t a(V_s)ds + W_t$. W is an l-dim $\mathbb{G}$-Brownian motion independent of B, and a is a smooth and bounded with $a(K) = 0$. Z is an abstract process modelling information contained in security prices.
- *Market information* is $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^N \vee \mathbb{F}^Z \vee \mathbb{F}^D$; $\mathbb{F}^Z \vee \mathbb{F}^D$ will be termed *background filtration*.

Pricing basic corporate securities and filtering

- Market uses risk-neutral pricing wrt $\mathbb{F}^M \Rightarrow$ ex-dividend price of a security with cash flow stream $(H_t)_{0 \le t \le T}$ is

$$\Pi_t^H = E^Q\big(\int_t^T e^{-r(s-t)} dH_s \mid \mathcal{F}_t^M \big) , \quad t \le T. \tag{8}$$

- Consider now a basic corporate securities with $\mathbb{F}^N \vee \mathbb{F}^D$-adapted cash flows such as bonds, CDS or the equity value of the firm (stock price). Iterated conditional expectations gives for the pre-default value of such a security

$$1_{\{\tau > t\}} \Pi_t^H = E^Q\Big(E^Q\big(1_{\{\tau > t\}} \int_t^T e^{-r(s-t)} dH_s \mid \mathcal{G}_t\big) \mid \mathcal{F}_t^M \Big).$$

Basic corporate securities and filtering

- Markov property of V implies that for basic corporate securities the inner conditional expectation is of the form $1_{\{\tau>t\}}h(t, V_t)$, the so-called *full-information value*. Hence

$$1_{\{\tau>t\}}\Pi_t^H = 1_{\{\tau>t\}}E^Q\big(h(t, V_t) \mid \mathcal{F}_t^M\big). \tag{9}$$

- Evaluation of this expression is a nonlinear filtering problem

Full-information value for debt securities

Two building blocks for the pricing of debt securities:

A survival claim has payoff $1_{\{\tau > T\}}$. Corresponding full-information value h^{surv} solves

$$\frac{dh^{\text{surv}}}{dt} + \mathcal{L}_V h^{\text{surv}} = r h^{\text{surv}}, \quad h^{\text{surv}}(t, K) = 0, t \leq T, \; h^{\text{surv}}(T, v) = 1.$$

Here $\mathcal{L}_V f = (r - \lambda^D \bar{\delta}) v \frac{df}{dv} + \frac{1}{2}\sigma^2 v^2 \frac{d^2 f}{dv^2}$ is the generator of V.

A payment-at-default claim pays one unit directly at τ, provided that $\tau \leq T$. h^{def} solves

$$\frac{dh^{\text{def}}}{dt} + \mathcal{L}_V h^{\text{def}} = r h^{\text{def}}, \quad h^{\text{def}}(t, K) = 1, t \leq T, \; h^{\text{def}}(T, v) = 0.$$

h^{surv} and h^{def} can be computed using results for the first passage time of Brownian motion with drift

Equity Pricing

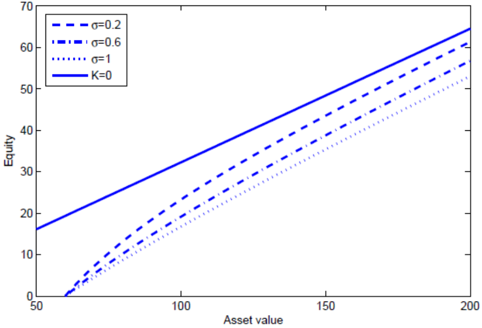
The *equity value* is defined as value of dividend payments up to default time time τ. Full-information value $S(v)$ satisfies

$$S(v) = E^Q\Big(\int_0^\tau e^{-rs} dD_s \mid V_0 = v\Big) = E^Q\Big(\int_0^\tau e^{-rs}\overline{\delta} V_s \lambda^D\, ds \mid V_0 = v\Big). \tag{10}$$

Hence S is time-independent and solves $\mathcal{L}_V S(v) + \overline{\delta}\lambda^D v = rS(v)$, with boundary condition $S(K) = 0$. For $K = 0$ (and hence $\tau = \infty$) we get $S(v) = v$. For $K > 0$ one has

Proposition. The full information value (10) of the firms equity is given by $S(v) = v - K(\frac{v}{K})^{\alpha^*}$, where α^* is the negative root of the equation $(r - \lambda^D\overline{\delta})\alpha + \frac{1}{2}\sigma^2\alpha(\alpha - 1) - r = 0$.

Equity Pricing ctd



Value of the firm's equity as function of the asset value for different σ and with $K = 60$. The straight line is the equity value for $K = 0$.

The filtering problem

Recall that we want to compute recursively the conditional expectation

$$1_{\{\tau > t\}} E^Q\big(f(V_t) \mid \mathcal{F}_t^M\big),\ t \leq T,\ f \in L^\infty\big([K, \infty)\big)\,. \tag{11}$$

This is a *nonstandard filtering problem*, due to inclusion of default history $\mathbb{F}^N$ in the observation filtration:

- Under full information $\mathbb{G}$ τ is a predictable stopping time and does not admit an intensity.
- In standard filtering theory with point process information on the other hand N is assumed to have a $\mathbb{G}$ intensity.

Basic idea. Reduce (11) to a filtering problem wrt background filtration $\mathbb{F}^Z \vee \mathbb{F}^D$ via Dellacherie formula.

Reduction to background filtration

Using the Dellacherie formula, we get

$$1_{\{\tau>t\}} E^Q\big(f(V_t) \mid \mathcal{F}_t^M\big) = 1_{\{\tau>t\}} \frac{E^Q\big(f(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{Q\big(\tau > t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}\,.$$

Denote by V^τ the process $V_t^\tau = V_{t\wedge\tau}$. By definition of τ we have $\{\tau > t\} = \{V_t^\tau > K\}$; moreover, $V_t = V_t^\tau$ for $t \le \tau$. Hence we get

$$1_{\{\tau>t\}} E^Q\big(f(V_t) \mid \mathcal{F}_t^M\big) = 1_{\{\tau>t\}} \frac{E^Q\big(f(V_t^\tau) 1_{\{V_t^\tau>K\}} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{Q\big(V_t^\tau > K \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}\,. \tag{12}$$

Remark. (12) is a filtering problem with standard diffusion and point process information. On the other hand new signal process $X := V^\tau$ with state space S^X is a *stopped diffusion process.*

Measure transform

Start with independent processes (X, Z) on $(\Omega, \mathcal{G}, Q^*)$ such that X is a stopped geometric Brownian motion and Z is a standard BM. (We largely ignore dividend payments.)
Consider the density martingale $L_t = \frac{dQ}{dQ^*}_{|\mathcal{F}_t}$ with

$$L_t = \exp\Big(\int_0^t a(X_s)^\top dZ_s - \frac{1}{2}\int_0^t |a(X_s)|^2\, ds \Big)\,. \qquad (13)$$

Girsanov $\Rightarrow$ the pair (X, Z) has the right law under Q and we have

$$E^Q\big(f(X_t) \mid \mathcal{F}_t^Z\big) = \frac{E^{Q^*}\big(f(X_t)L_t \mid \mathcal{F}_t^Z\big)}{E^{Q^*}\big(L_t \mid \mathcal{F}_t^Z\big)} =: \frac{\Sigma_t f}{\Sigma_t 1}\,. \qquad (14)$$

SPDE for the density of Σ_t

Suppose that Σ_t is absolutely continuous with density u. u is related to the SPDE

$$du(t) = \mathcal{L}^* u(t)dt + a^\top u(t)dZ_t, \quad u(0) = \pi_0.$$

Formal interpretation. Denote by (f, g) the scalar product on $L^2(S^X)$. Then u is an $\mathbb{F}^Z$ adapted continuous process with values in the Sobolev space $H_1^0(S^X) \cap H^2(S^X)$, and one has for $v \in L^2$

$$\big(u(t), v\big) = \big(u(0), v\big) + \int_0^t (\mathcal{L}^* u(s), v)ds + \int_0^t \big(a^\top u(s), v\big) dZ_s. \tag{15}$$

Theorem. ([Pardoux, 1978]) There is a unique solution u of equation (15). Moreover, for $f \in L^\infty(S^X)$,

$$\Sigma_t f = \big(u(t), f\big) + \nu_K(t) f(K) \text{ where } \nu_K(t) = \int_0^t \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s, K)ds\,. \tag{16}$$

Comments and Implications

Comments.

- The measure Σ_t consists of two parts: an absolutely continuous part with density $u(t)$ and a point mass $\nu_K(t)$ at the boundary K.
- Boundary term drives form of default intensities.
- We give a simplified presentation here: the result has been shown only for bounded domain $[K, N)$ (see paper).
- Numerical solution either via Galerkin approximation of (15) or via Markov-chain approximation of X.

Corollary. We get for the original filtering problem

$$E(f(V_t) \mid \mathcal{F}_t^M) = \big(\tilde{\pi}(t), f\big) \text{ with } \tilde{\pi}(t,x) = \frac{u(t,x)}{\big(u(t),1\big)}. \qquad (17)$$

Default Intensity

Theorem. The $\mathbb{F}^M$ compensator of N_t is given by $(\Lambda_{t\wedge\tau})_{t\geq 0}$ where

$$\Lambda_t = \int_0^t \lambda_s ds \text{ with } \lambda_t = \frac{1}{2}\sigma^2 K^2 \frac{d\pi}{dx}(t,K)\,. \tag{18}$$

Here $\pi(t,x)$ is conditional density of X_t given $\mathcal{F}_t^M$.

- This extends earlier results of [Duffie and Lando, 2001] and [Frey and Schmidt, 2009] to the case where information arrives continuously.
- An alternative characterization of the compensator of N has recently been given by [Cetin, 2011].

Filter equations

Theorem. For $f \in C^{1,2}([0,T] \times S^X)$ the projection $\widehat{f}_t = E(f(t,X_t) \mid \mathcal{F}_t^M)$ has dynamics

$$\widehat{f}_t = \widehat{f}_0 + \int_0^t (\widehat{\frac{df}{dt}})_s + (1 - N_{s-})(\widehat{\mathcal{L}_X f})_s \, ds + \int_0^{t\wedge\tau} (\widehat{fa^\top})_s - \widehat{f}_s \widehat{a^\top}_s \, dM_s^Z$$

$$+ \int_0^{t\wedge\tau} (f(s,K) - \widehat{f}_{s-}) \, d(N_s - \lambda_s ds)$$

$$+ \int_0^{t\wedge\tau} \int_{\mathbb{R}^+} \frac{(\widehat{f\varphi_d(y)})_{s-} - \widehat{f}_{s-}(\widehat{\varphi_d(y)})_{s-}}{\widehat{f}_{s-}(\widehat{\varphi_d(y)})_{s-}} (\mu^D - \gamma^D)(dy, ds) \, .$$

Here $M^Z = Z_t - \int_0^t \widehat{a}_s ds$ is a $\mathbb{F}^M$ Brownian motion and $(\mu^D - \gamma^D)(dy, ds)$ is the $\mathbb{F}^M$-compensated random measure associated with the dividends.

Stock price dynamics

Using the filter equations it is straightforward to compute the semimartingale decomposition of the stock price $\widehat{S}_t$. We get

$$d\widehat{S}_t = (1 - N_{t-})(r\widehat{S}_t - \lambda^D \bar{\delta} \widehat{V}_t)dt + (1 - N_{t-})\big((\widehat{Sa^\top})_t - \widehat{S}_t \widehat{a^\top}_t\big)\, dM_t^Z$$
$$- (1 - N_{t-})\widehat{S}_{t-}\, d(N_t - \lambda_t dt) + \text{integral wrt}(\mu^D - \gamma^D)(dy, dt)$$

- Similar formulas can be obtained for debt securities.
- Note that stock-price dynamics can be quite wild even if asset price dynamics follow standard geometric Brownian motion.

Derivative Pricing

Basic corporate securities ($\mathbb{F}^D \vee \mathbb{F}^N$-adapted payoff H).
Here $\Pi_t^H = 1_{\{\tau > t\}}(\pi(t), h)$. Note that price is *linear* in $\pi(t)$

Options. Here payoff depends on price path of basic corporate securities; examples include equity options, convertibles, ...
Consider an equity option with payoff $f(\widehat{S}_T)$. Price is given by

$$\Pi_t^f = E^Q\left(f(\widehat{S}_T) \mid \mathcal{F}_t^M\right) = E^Q\left(f\left(\frac{(u(T),S)}{(u(t),1)}\right) \mid \mathcal{F}_t^M\right) = C(t, \pi(t))$$

for an appropriate function $C(\cdot)$. In this case the form of $C(\cdot)$ depends on dynamics of $\pi(t)$.

Note that incomplete-information model is a factor model with (infinite dimensional) factor process $\pi(t)$ (or $u(t)$).

Sampling of a stock price trajectory

Computation of $C(\cdot)$ is best done via Monte Carlo.

Algorithm. (Sampling of a stock price trajectory)

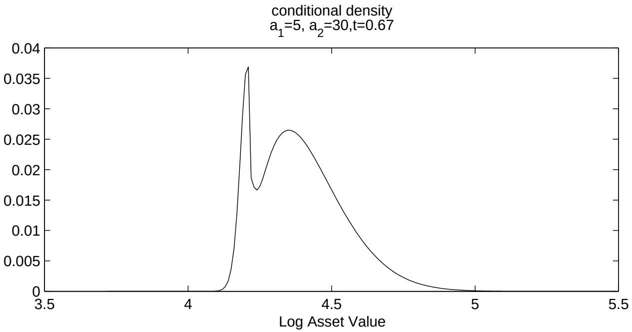
1. Generate a trajectory $(V_t)_{t=0}^T$ and associated trajectory of N.
2. Generate noisy observation $(Z_t)_{t=0}^T$, using trajectory of V from 1. as input.
3. Compute (numerically) the solution $\tilde{u}(t)$ of the Zakai SPDE (15) for the given observation.
4. Return $\widehat{S}_t = (1 - N_t)(\tilde{u}(t), S)/(\tilde{u}(t), 1)$, $0 \leq t \leq T$.

In Step 2 one can alternatively use Markov chain approximations to compute $\widehat{S}_t$.

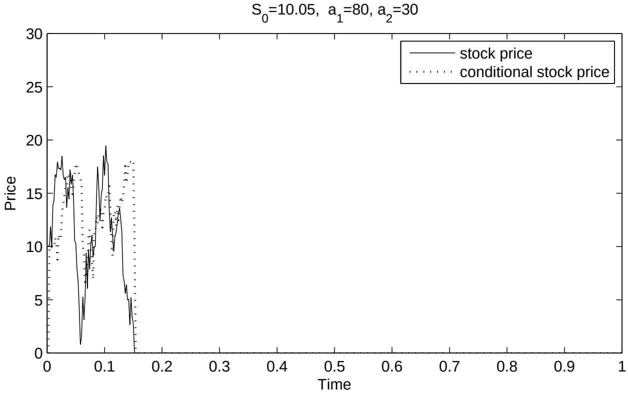
Calibration

1. We view Z as abstract process that models information contained in prices of traded corporate securities. Hence the current value $\pi(t)$ of the density of the asset value process is unobservable and investors need to determine $\pi(t)$ by "matching" market and model prices.
   - Large freedom for calibration
   - If we approximate $\pi(t) \approx \sum_{i=1}^{m} \psi_i e_i(t)$ (Galerkin method), calibration to prices of basic corporate securities is a linear or convex programming problem for coefficients $\psi_1, \dots, \psi_m$.
2. Determination of model parameters, in particular σ and parameters of $a(\cdot)$ is future research)

Numerical Experiments



A stock price path



Bain, A. and Crisan, D. (2009).
Fundamentals of Stochastic Filtering.
Springer, New York.

Brémaud, P. (1981).
Point Processes and Queues: Martingale Dynamics.
Springer, New York.

Budhiraja, A., Chen, L., and Lee, C. (2007).
A survey of numerical methods for nonlinear filtering problems.

Physica D, 230:27–36.

Cetin, U. (2011).
On absolutely continuous compensators and nonlinear filtering equations in default risk models.
preprint, London School of Economics.

Coculescu, D., Geman, H., , and Jeanblanc, M. (2008).

Valuation of default sensitive claims under imperfect information.
Finance & Stochastics, 12:195–218.

Collin-Dufresne, P., Goldstein, R., and Helwege, J. (2003).
Is credit event risk priced? modeling contagion via the updating of beliefs.
Preprint, Carnegie Mellon University.

Davis, M. and Marcus, S. (1981).
An introduction to nonlinear filtering.
In Hazewinkel, M. and Willems, J., editors, *Stochastic Systems: The Mathematics of Filtering and Identification*, pages 53–75. D. Reidel.

Duffie, D., Eckner, A., Horel, G., and Saita, L. (2009).
Frailty correlated default.
Journal of Finance, 64:2089–2123.

Duffie, D. and Lando, D. (2001).
Term structure of credit risk with incomplete accounting observations.
Econometrica, 69:633–664.

Frey, R. and Lu, D. (2012).
Dynamics of corporate security prices in firm value models with incomplete information.
preprint, Institute of Statistics and Mathematics, WU Vienna.

Frey, R. and Runggaldier, W. (2008).
Nonlinear filtering in models for interest-rate and credit risk.
preprint, to appear in *Handbook of Nonlinear Filtering*.

Frey, R. and Runggaldier, W. (2010).
Pricing credit derivatives under incomplete information: a nonlinear filtering approach.
Finance and Stochastics.

Frey, R. and Schmidt, T. (2009).
Pricing corporate securities under noisy asset information.
Mathematical Finance, 19:403 – 421.

Frey, R. and Schmidt, T. (2010).
Filtering and incomplete information in credit risk.
In et. al., T. B., editor, *Recent Advancements in the Theory and Practice of Credit Derivatives (to appear)*. Bloomberg Press.

Frey, R. and Schmidt, T. (2012).
Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering.
Finance and Stochastics.

Giesecke, K. and Goldberg, L. (2004).
Sequential defaults and incomplete information.
Journal of Risk, 7:1–26.

Jarrow, R. and Protter, P. (2004).
Structural versus reduced-form models: a new information based perspective.
Journal of Investment management, 2:1–10.

Pardoux, E. (1978).
Filtrage de diffusions avec condition frontières, volume 636 of *Lecture Notes in Mathematics*, pages 163–188.
Springer, Berlin.

Schönbucher, P. (2004).
Information-driven default contagion.
Preprint, Department of Mathematics, ETH Zürich.