Weak Maximum Principle for Stochastic Control

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Outline of Talk

(i) Introduction

(ii) Model and main results

(iii) Two examples

(iv) Key steps in proofs

DPP and SMP

- There has been extensive research in stochastic control theory. Two principal and most commonly used methods are dynamic programming principle (DPP) and stochastic maximum principle (SMP)
- DPP leads to Hamilton-Jacobi-Bellman nonlinear PDE, feedback control, viscosity solution, see Fleming-Rishel (1975), Fleming-Soner (2006),
- SMP leads to coupled forward backward SDE (FBSDE), see Yong-Zhou (1999).
- This talk is focused on SMP.

Literature on Necessary SMP

- Kushner (1965, 1972) is first to study necessary SMP.
- Bismut (1973, 1976, 1978), Bensoussan (1981) and Haussmann (1986) extend Kushner's SMP to more general stochastic control problems with control-free diffusion coefficients.
- Peng (1990) applies second order spike variation technique to derive necessary SMP to stochastic control problems with controlled diffusion coefficients. Zhou (1991) simplifies Peng's proof.
- Tang-Li (1994) extends Peng's SMP to systems with jump diffusions and Cadenillas-Karatzas (1995) with random coefficients.

Literature on Sufficient SMP

- Bismut (1978) is first to investigate sufficient SMP.
- Zhou (1996) proves that Peng's SMP is also sufficient in presence of certain convexity condition.
- Framstad-Øksendal-Sulem (2004) extends sufficient SMP to systems with jump diffusion,
- Donnelly (2011) with Markovian regime-switching diffusion and
- Zhang-Elliott-Siu (2012) with Markovian regime-switching jump diffusion.

Essence of SMP

- Necessary SMP states that any optimal control along with optimal state trajectory must solve a system of forward-backward SDE plus a maximum condition of optimal control on Hamiltonian.
- Necessary condition together with certain concavity conditions on Hamiltonian give sufficient condition of optimality.
- Major difficulty of generalizing classical Pontryagin's maximum principle for deterministic optimal control to a stochastic control problem with controlled diffusion term is that, in some cases, Hamiltonian is a convex function of control variable and achieves minimum at optimal control.
- One of major contributions of Peng's SMP is introduction of modified Hamiltonian and second order adjoint stochastic processes.
- When Hamiltonian is convex, second order term turns modified Hamiltonian to a concave function which achieves maximum at optimal control.

- Modified Hamiltonian and second order adjoint equation are introduced to preserve maximality condition of Pontryagin's maximum principle.
- One has to assume that all functions involved are twice continuously differentiable in state variable in order to use second order variation, which limits scope of problems applicable to theorem.
- One has to solve associated second order adjoint BSDE with dimensionality equal to square of that of its first order counterpart, which makes problem more difficult to solve, at least numerically.
- One can not get sufficient condition by enhancing necessary condition with some joint concavity condition to modified Hamiltonian and instead one has to add some joint concavity condition to Hamiltonian, which illustrates that necessary SMP is not completely compatible with sufficient SMP.

Model

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. Suppose that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous natural filtration generated by $W(t)$, an m-dimensional standard Brownian motion, augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$.

- We consider a stochastic control model where state of system is governed by a controlled SDE

$$\begin{cases} dx(t) & = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \ t \in [0, T], \\ x(0) & = x_0 \end{cases} \tag{1}$$

where u is a $\mathbb{R}^k$ valued progressively measurable (with respect to $\{\mathcal{F}_t\}_{t\geq 0}$) process, and $T > 0$ is a fixed finite time horizon, $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, $\sigma = (\sigma^1, \ldots, \sigma^m) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to (\mathbb{R}^n)^m$ are given functions.

- A control u is called *admissible* if it is a progressively measurable process, valued in U, a nonempty closed convex subset of $\mathbb{R}^k$, such that

$$\|u\| := \left(\int_0^T E|u(t)|^4 dt \right)^{\frac{1}{4}} < \infty. \tag{2}$$

Denote by $\mathcal{U}_{ad}$ set of all admissible controls.

Assumptions on b and σ

(A1) Maps b and σ are measurable, and there exists constants $L > 0$, $K > 0$ such that for $\phi = b$ and σ we have

$$\begin{cases} |\phi(t,x,u) - \phi(t,\hat{x},\hat{u})| \le L|x-\hat{x}| + K|u-\hat{u}|, \\ \forall t \in [0,T],\ x,\hat{x} \in \mathbb{R}^n,\ u,\hat{u} \in \mathbb{R}^k, \\ |\phi(t,0,u)| \le L + K,\ \forall (t,u) \in [0,T] \times \mathbb{R}^k. \end{cases}$$

(A2) Maps b and σ are C^1 in x. Moreover, there exist constants $L > 0$ and a modulus of continuity $\bar{\omega} : [0,+\infty) \to [0,+\infty)$ such that for $\phi = b$ and σ, we have

$$\begin{cases} |\phi_x(t,x,u) - \phi_x(t,\hat{x},\hat{u})| \le L|x-\hat{x}| + \bar{\omega}(d(u,\hat{u})), \\ \forall t \in [0,T],\ x,\hat{x} \in \mathbb{R}^n,\ u,\hat{u} \in \mathbb{R}^k. \end{cases}$$

Optmal Stochastic Control Problem

- Cost Functional:

$$J(u) = E\left[\int_0^T f(t, x(t), u(t))dt + h(x(T))\right], \qquad (3)$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ are given functions.

- **Problem (S)** Minimize (3) over $\mathcal{U}_{ad}$.

- Any $\bar{u} \in \mathcal{U}_{ad}$ satisfying

$$J(\bar{u}) = \inf_{u \in \mathcal{U}_{ad}} J(u)$$

is called an *optimal control*. Corresponding $\bar{x}$ and $(\bar{x}, \bar{u})$ are called an *optimal state process* and *optimal pair*, respectively.

Assumptions on f and h

(A3) Maps f and h are measurable and there exists constants $K_1, K_2 \geq 0$ such that

$$|f(t,x,u) - f(t,x,\hat{u})| \leq (K_1 + K_2(|x| + |u| + |\hat{u}|))\,|u - \hat{u}|.$$

(A4) Maps f and h are C^1 in x. Moreover, there exist constants $L > 0$ and a modulus of continuity $\bar{\omega} : [0,+\infty) \to [0,+\infty)$ such that for $\phi = f$ and h, we have

$$\begin{cases} |\phi_x(t,x,u) - \phi_x(t,\hat{x},\hat{u})| \leq L|x - \hat{x}| + \bar{\omega}(d(u,\hat{u})), \\ \forall t \in [0,T],\ x,\hat{x} \in \mathbb{R}^n,\ u,\hat{u} \in \mathbb{R}^k \\ |\phi_x(t,0,u)| \leq L + K,\ \forall (t,u) \in [0,T] \times \mathbb{R}^k. \end{cases}$$

Remark 1. Assumptions **(A3)** and **(A4)** together cover many cases, including all quadratic functions in x and u. For instance, if f is Lipschitz in u, then $K_2 = 0$. On other hand, if f is differentiable with respect to u and f_u satisfies a linear growth condition in u, then K_2 is a positive constant.

Hamiltonian and Adjoint Equation

- Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \longrightarrow \mathbb{R}$, for stochastic control problem (1) and (3), is defined by
$$H(t,x,u,p,q) = \langle p, b(t,x,u)\rangle + tr[q^T \sigma(t,x,u)] - f(t,x,u). \tag{4}$$
- Given an admissible pair (x,u), *adjoint equation* is following linear BSDE:
$$\begin{cases} dp(t) &= -H_x(t,x(t),u(t),p(t),q(t))dt + q(t)dW(t), \;\; t \in [0,T] \\ p(T) &= -h_x(x(T)). \end{cases} \tag{5}$$
- Any pair of stochastic processes $(p,q) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times (L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)^m)$ is called an *adapted solution* of (5), where $q = (q^1,\ldots,q^m)$ and $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is space of $\mathbb{R}^n$-valued square integrable progressively measurable processes.
- Under assumptions **(A1)**-**(A4)**, for any $(x,u) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times \mathcal{U}_{ad}$, (5) admits a unique adapted solution (p,q).
- If $(\bar{x},\bar{u})$ is an optimal (resp. admissible) pair, and $(\bar{p},\bar{q})$ is adapted solution of (5), then $(\bar{x},\bar{u},\bar{p},\bar{q})$ is called an *optimal* (resp. *admissible) 4-tuple.*

Second Order Adjoint Equation(Yong-Zhou (1999))

- Second order adjoint equation: $P, Q_j \in L^2_{\mathcal{F}}(0,T;\mathbb{S}^n)$ for $j = 1, \ldots, m$, satisfy following BSDE

$$
\begin{aligned}
dP(t) \;=\; & -\Big\{ b_x(t,x(t),u(t))^T P(t) + P(t) b_x(t,x(t),u(t)) \\
& + \sum_{j=1}^m \sigma_x^j(t,x(t),u(t))^T P(t) \sigma_x^j(t,x(t),u(t)) \\
& + \sum_{j=1}^m \{\sigma_x^j(t,x(t),u(t))^T Q_j(t) + Q_j(t)\sigma_x^j(t,x(t),u(t))\} \\
& + H_{xx}(t,x(t),u(t),p(t),q(t)) \Big\} + \sum_{j=1}^m Q_j(t) dW^j(t) \\
P(T) \;=\; & -h_{xx}(x(T)).
\end{aligned}
$$

Necessary SMP

Theorem 2. *(Peng (1990)) Assume technical conditions. Let* $(\bar{x}, \bar{u})$ *be an optimal pair for Problem (S). Then there are pairs* (p, q) *and* (P, Q) *satisfying the first-order and second-order adjoint equations, such that*

$$\mathcal{H}(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \mathcal{H}(t, \bar{x}(t), u), \quad a.e.\ t \in [0, T],\ P\ a.s.$$

where the modified Hamiltonian $\mathcal{H}$ *is defined by,*

$$\begin{aligned} \mathcal{H}(t, x, u) &= H(t, x, u, p(t), q(t)) \\ &\quad + \frac{1}{2} \mathrm{tr} \left(\sigma(t, x, u)^T P(t) (\sigma(t, x, u) - 2\sigma(t, \bar{x}(t), \bar{u}(t)) \right) \end{aligned}$$

Remark 3. Ideally, we would like to have the following maximum principle:

$$H(t, \bar{x}(t), \bar{u}(t), p(t), q(t)) = \max_{u \in U} H(t, \bar{x}(t), u, p(t), q(t)).$$

This is IMPOSSIBLE. There are counter-examples to it (Yong-Zhou (1999)).

Clarke's Generalized Gradient

We recall some basic concepts and properties in nonsmooth analysis and optimization, which are needed in main results.

Definition 4. *(Generalized directional derivative) Let C be an open subset of a Banach space X, and let a function $f : C \longrightarrow \mathbb{R}$ be given. We suppose that f is Lipschitz near $x \in C$. Generalized directional derivative of f at x in direction v, denoted $f^o(x; v)$, is given by*

$$f^o(x; v) = \limsup_{y \to_C x, \lambda \downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 5. *(Clarke's generalized gradient) Let X^* denote dual of X and $\langle \cdot, \cdot \rangle$ be duality pairing between X and X^*. Generalized gradient of f at x, denoted $\partial f(x)$, is set of all ζ in X^* satisfying*

$$f^o(x; v) \geq \langle v, \zeta \rangle \text{ for } \forall v \in X.$$

Theorem 6. *If f attains a local minimum or maximum at x, then*

$$0 \in \partial f(x).$$

Example. $f(x) = |x|$ then $\partial f(0) = [-1, 1]$.

Clarke's Tangent Cone and Normal Cone

Definition 7. *(Distance function) Let X be a Banach space and C be a non-empty subset of X. Distance function $d_C : X \to \mathbb{R}$ is defined as*

$$d_C(x) = \inf\{\|x - c\| : c \in C\}.$$

Definition 8. *(Tangent cone) Suppose $x \in C$. A vector v in X is a tangent to C at x provided $d_C^o(x; v) = 0$. Tangent cone to C at x, denoted as $T_C(x)$, is set of all tangents to C at x.*

Definition 9. *(Normal cone) Let $x \in C$. Normal cone to C at x is defined by polarity with $T_C(x)$:*

$$N_C(x) = \{\xi \in X^* : \langle \xi, v \rangle \leq 0 \text{ for all } v \in T_C(x)\}.$$

Theorem 10. *Assume that f is Lipschitz near x and attains a minimum over C at x. Then*

$$0 \in \partial f(x) + N_C(x).$$

Main Result

Theorem 11. *(Weak Necessary SMP) Let assumptions **(A1)-(A4)** hold. Let $(\bar{x}, \bar{u})$ be an optimal pair of Problem (S). Then there exists a pair of stochastic processes $(\bar{p}, \bar{q}) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)^m)$ satisfying adjoint equation* (5)*, such that*

$$0 \in \partial_u(-H)(t, \bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t)) + N_U(\bar{u}(t)), \;\; a.e. \; t \in [0, T], \; \mathbb{P}\text{-}a.s. \tag{6}$$

Theorem 12. *(Weak Sufficient SMP) Let assumptions **(A1)-(A4)** hold and let $(\bar{x}, \bar{u}, \bar{p}, \bar{q})$ be an admissible 4-tuple satisfying (6). Suppose further that h is convex and $H(t, \cdot, \cdot, \bar{p}(t), \bar{q}(t))$ is concave for all $t \in [0, T]$ a.s. Then $(\bar{x}, \bar{u})$ is an optimal pair for Problem (S).*

Example 1

Consider following stochastic control problem (Yong-Zhou (1999), Example 3.5.3):

$$\begin{cases} dx(t) = u(t)dW(t), t \in [0,1] \\ x(0) = 0 \end{cases} \tag{7}$$

with control constraint set $U = [0,1]$ and cost functional

$$J(u) = E\left\{-\int_0^1 u(t)dt + \frac{1}{2}x(1)^2\right\}.$$

Suppose $(\bar{x}, \bar{u})$ is an optimal pair, then corresponding adjoint equation is

$$\begin{cases} d\bar{p}(t) = \bar{q}(t)dW(t), t \in [0,1] \\ \bar{p}(1) = -\bar{x}(1). \end{cases} \tag{8}$$

Using (7) and (8) and via a simple calculation we obtain

$$\bar{p}(t) = -\int_0^t \bar{u}(s)dW(s) - \int_t^1 \left(\bar{u}(s) + \bar{q}(s)\right) dW(s).$$

Since adjoint process $\bar{p}(t)$ is adapted to filtration $\mathcal{F}_t$, we must have

$$\bar{u}(t) + \bar{q}(t) = 0 \ \textit{for all} \ t \in [0,1], \mathbb{P}\text{-}a.s. \tag{9}$$

Corresponding Hamiltonian is

$$H(t, x, u, \bar{p}(t), \bar{q}(t)) = \bar{q}(t)u + u.$$

Since problem satisfies **(A1)-(A4)**, by Theorem 11 and (6), we have

$$0 \in -(\bar{q}(t) + 1) + N_{[0,1]}(\bar{u}(t)) \textit{ for all } t \in [0, 1], \mathbb{P}\text{-}a.s.$$

Consequently, on any nonzero measurable set $E \in S = \Omega \times [0, 1]$, we can only have following three cases:

Case 1 : $0 < \bar{u}(t) < 1 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = \{0\}$ *and* $\bar{q}(t) = -1$.

Case 2 : $\bar{u}(t) = 0 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = (-\infty, 0]$ *and* $\bar{q}(t) + 1 \leq 0$.

Case 3 : $\bar{u}(t) = 1 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = [0, +\infty)$ *and* $\bar{q}(t) + 1 \geq 0$.

Suppose Case 1 or Case 2 is true, then $\bar{u}(t) + \bar{q}(t) \leq \bar{u}(t) - 1 < 0$ for some nonzero measurable set $E \in S$, contradiction to (9). Hence, we have $\bar{u}(t) = 1$ for every $t \in [0, 1], \mathbb{P}$-a.s. and $\bar{x}(t) = W(t)$ and $(\bar{p}(t), \bar{q}(t)) = (-W(t), -1)$ for $t \in [0, 1]$. Since $(x, u) \mapsto H(t, x, u, \bar{p}(t), \bar{q}(t)) = -u+u = 0$ is concave and $x \mapsto h(x) = \frac{1}{2}x^2$ is convex, we conclude that $\bar{u}(t) = 1$ is optimal control using Theorem 12.

Example 2

Consider following stochastic control problem (Yong-Zhou (1999), Example 3.3.1):

$$\begin{cases} dx(t) = u(t)dW(t), t \in [0,1] \\ x(0) = 0 \end{cases} \tag{10}$$

with control constraint set $U = [-1, 1]$ and cost functional

$$J(u) = E\left\{ \int_0^1 [x(t)^2 - \frac{1}{2}u(t)^2]dt + x(1)^2 \right\}.$$

Suppose $(\bar{x}, \bar{u})$ is an optimal pair, then corresponding adjoint equation is

$$\begin{cases} d\bar{p}(t) = 2\bar{x}(t)dt + \bar{q}(t)dW(t), t \in [0,1] \\ \bar{p}(1) = -2\bar{x}(1). \end{cases} \tag{11}$$

Using (10), (11) and via a simple calculation, we obtain

$$\bar{p}(t) = -2\int_0^t (2-t)\bar{u}(s)dW(s) - \int_t^1 ((4-2s)\bar{u}(s) + \bar{q}(s))dW(s).$$

Since adjoint process $\bar{p}(t)$ is adapted to filtration $\mathcal{F}_t$, we must have

$$(4-2t)\bar{u}(t) + \bar{q}(t) = 0 \ \textit{for all} \ t \in [0,1], \mathbb{P}\text{-}a.s. \tag{12}$$

Corresponding Hamiltonian is

$$H(t, x, u, \bar{p}(t), \bar{q}(t)) = \bar{q}(t)u - x^2 + \frac{1}{2}u^2.$$

Since problem satisfies assumptions **(A1)**-**(A4)**, by Theorem 11 and (6), we have

$$0 \in -\left(\bar{q}(t) + \bar{u}(t)\right) + N_{[-1,1]}(\bar{u}(t)) \text{ \textit{for all} } t \in [0,1], \mathbb{P}\text{-}a.s.$$

Consequently, on any nonzero measurable set $E \in S$, we can only have following three cases:

Case 1 $-1 < \bar{u}(t) < 1 \Longrightarrow N_{[-1,1]}(\bar{u}(t)) = \{0\}$ and $\bar{q}(t) + \bar{u}(t) = 0$.

Case 2 $\bar{u}(t) = -1 \Longrightarrow N_{[-1,1]}(\bar{u}(t)) = (-\infty, 0]$ and $\bar{q}(t) + \bar{u}(t) \leq 0$.

Case 3 $\bar{u}(t) = 1 \Longrightarrow N_{[-1,1]}(\bar{u}(t)) = [0, +\infty)$ and $\bar{q}(t) + \bar{u}(t) \geq 0$.

Suppose Case 2 were true. Then

$$(4 - 2t)\bar{u}(t) + \bar{q}(t) \leq -3 + 2t \leq -1.$$

Contradiction to (12). Suppose Case 3 were true. Then

$$(4 - 2t)\bar{u}(t) + \bar{q}(t) \geq 3 - 2t \geq 1.$$

Contradiction to (12). Hence we have

$$-1 < \bar{u}(t) < 1 \text{ and } \bar{q}(t) + \bar{u}(t) = 0 \text{ for all } t \in [0,1], \mathbb{P}\text{-}a.s.$$

By (12), (10) and (11), we obtain $(\bar{x}(t), \bar{u}(t)) = (0,0)$ and $(\bar{p}(t), \bar{q}(t)) = (0,0)$. However, since $(x,u) \mapsto H(t,x,u,\bar{p}(t),\bar{q}(t)) = -x^2 + \frac{1}{2}u^2$ is not a concave function, Theorem 12 cannot be used. We have to use other method to check optimality of $\bar{u}$. Substituting $x(t) = \int_0^t u(s)dW(s)$ into cost functional and via simple calculations, we obtain

$$J(u) = E\int_0^1 \left(\frac{3}{2} - t\right) u(t)^2 dt$$

Hence, we conclude that $\bar{u}(t) = 0$ is indeed optimal control.

Remark 13. Note that Hamiltonian

$$u \mapsto H(t, \bar{x}(t), u, \bar{p}(t), \bar{q}(t)) = \frac{1}{2}u^2$$

is a convex function and $\bar{u}(t) = 0$ is a minimum point. This is reason that Peng (1990) and Yong-Zhou (1999) introduce the modified Hamiltonian ($\mathcal{H}$-function)

$$u \mapsto \mathcal{H}(t, \bar{x}(t), u) = \frac{1}{2}(2t-3)u^2$$

which is a *concave* function and makes $\bar{u}(t) = 0$ a maximum point.

Moment Estimate

We prove some preliminary results, which will be useful in sequel. Hereafter, K represents a generic constant. Following result is a simplified version of Yong-Zhou (1999), Lemma 3.4.2.

Lemma 14. *Let $Y(t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ be solution of following:*

$$\begin{cases} dY(t) &= \{A(t)Y(t) + \alpha(t)\}dt + \displaystyle\sum_{j=1}^{m}\{B^j(t)Y(t) + \beta^j(t)\}dW^j(t), \\ Y(0) &= Y_0, \end{cases}$$

where $A, B^j : [0,T] \times \Omega \longrightarrow \mathbb{R}^{n\times n}$ and $\alpha, \beta^j : [0,T] \times \Omega \longrightarrow \mathbb{R}^n$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, and

$$\begin{cases} |A(t)|, |B^j(t)| \leq L, \ a.e.t \in [0,T], \ \mathbb{P} - a.s., \ 1 \leq j \leq m, \\ \displaystyle\int_0^T E|\alpha(s)|^{2k} \, \mathrm{d}s + \int_0^T E|\beta^j(s)|^{2k} ds < \infty, \ 1 \leq j \leq m, \end{cases} \tag{13}$$

for some $k \geq 1$. Then

$$\sup_{t\in[0,T]} E|Y(t)|^{2k} \leq K \left\{ E|Y_0|^2 k + \int_0^T E|\alpha(s)|^{2k} ds + \sum_{j=1}^{m} \int E|\beta^j(s)|^{2k} ds \right\}. \tag{14}$$

Lipschitz Property

Lemma 15. *Let* $u_1, u_2 \in L^4(S; \mathbb{R}^k)$ *and* x_1, x_2 *be associated state processes satisfying* (1). *Then we have following inequality:*

$$\sup_{t \in [0,T]} E|x_1(t) - x_2(t)|^4 \leq K\|u_1 - u_2\|^4$$

Lemma 16. *Assume that* ***(A1)*** *holds. Given control* $u \in L^4(S; \mathbb{R}^k)$ *and associated state process* x *satisfying* (1), *then*

$$E\left(\sup_{0 \leq t \leq T} |x(t)|^2\right) \leq K\left(1 + |x_0|^2 + \int_0^T E|u(t)|^2 dt\right) \tag{15}$$

Lemma 17. *Cost functional* $J : L^4(S; \mathbb{R}^k) \to \mathbb{R}$ *is locally Lipschitz, i.e., for all* $\hat{u}$ *in* $L^4(S; \mathbb{R}^k)$ *there exists a small ball* $B_{\hat{u}}^M$ *with radius* $M > 0$ *containing* $\hat{u}$ *on which , we have*

$$|J(u_1) - J(u_2)| \leq K_{M,\hat{u}}\|u_1 - u_2\| \tag{16}$$

for $\forall u_1, u_2 \in B_{\hat{u}}^M$.

Let (x, u) be an admissible pair. Let $v \in Ł^4(S; \mathbb{R}^k)$ and $\epsilon > 0$. Define $u^\epsilon(t) \triangleq u(t) + \epsilon v(t)$ for all $t \in [0, T]$. Let (x^ϵ, u^ϵ) satisfy following stochastic control system:

$$\begin{cases} dx^\epsilon(t) &= b(t, x^\epsilon(t), u^\epsilon(t))dt + \sigma(t, x^\epsilon(t), u^\epsilon(t))dW(t), \ t \in [0, T], \\ x^\epsilon(0) &= x_0. \end{cases}$$

Next, for $\varphi = b, \ \sigma^j, \ (1 \leq j \leq m), \ f$, we define

$$\varphi_x(t) \triangleq \varphi_x(t, x(t), u(t)), \quad \delta\varphi(t) \triangleq \varphi(t, x(t), u^\epsilon(t)) - \varphi(t, x(t), u(t)).$$

Let y^ϵ be solution of following stochastic differential equation:

$$\begin{cases} dy^\epsilon(t) &= \{b_x(t)y^\epsilon(t) + \delta b(t)\} dt + \sum_{j=1}^{m} \left\{\sigma_x^j(t)y^\epsilon(t) + \delta\sigma^j(t)\right\} dW^j(t), \ t \in [0, T], \\ y^\epsilon(0) &= 0 \end{cases} \tag{17}$$

Remark 18. Variation in our proof is different from so-called spike variation technique in proof of Peng's maximum principle in Peng (1990) and Yong-Zhou (1999). In their proof, where $u^\epsilon(t) = u(t) + 1_{[\tau, \tau+\epsilon]}v(t)$, one first perturbs an optimal control on a small set of size ϵ and then let $\epsilon \to 0$. Whereas, in our proof we perturbs an optimal control over whole space.

Then reason behind this is that in definition of Clarke's generalized directional derivative, $v(t)$ represents a directional vector in $L^4(S;\mathbb{R}^k)$ and must be fixed. One perturbs control through multiplication of a scalar ϵ and letting $\epsilon \to 0$.

Following lemma gives Taylor expansion result of state process and cost functional.

Lemma 19. *Let assumptions **(A1)-(A4)** hold. Then, we have*

$$\sup_{t\in[0,T]} E\left|x^\epsilon(t) - x(t)\right|^2 = O(\epsilon^2), \tag{18}$$

$$\sup_{t\in[0,T]} E\left|y^\epsilon(t)\right|^2 = O(\epsilon^2), \tag{19}$$

$$\sup_{t\in[0,T]} E\left|x^\epsilon(t) - x(t) - y^\epsilon(t)\right|^2 = o(\epsilon^2). \tag{20}$$

Moreover, following expansion holds for cost functional:

$$\begin{aligned} J(u^\epsilon) = \; & J(u) + E\langle h_x(x(T)), y^\epsilon(t)\rangle \\ & + E\int_0^T \left\{\langle f_x(t, x(t), u(t)), y^\epsilon(t)\rangle + \delta f(t)\right\} dt + o(\epsilon). \end{aligned} \tag{21}$$

Duality Analysis

Lemma 20. *Let assumptions **(A1)-(A4)** hold. Let y^ϵ be solution of* (17) *and* (p, q) *be adapted solution of* (5). *Then*

$$E\langle p(T), y^\epsilon(T)\rangle = E\int_0^T \left\{\langle p(t), \delta b(t)\rangle + \langle f_x(t), y^\epsilon(t)\rangle + tr\left[q(t)^T \delta\sigma(t)\right]\right\} dt \tag{22}$$

Now we are able to give following lemma, which is of great importance.

Lemma 21. *Let assumptions **(A1)-(A4)** hold. For any* $\varepsilon > 0$ *and* $v \in L^4(S; \mathbb{R}^k)$, *define*

$$u^\epsilon(t) \triangleq u(t) + \epsilon v(t) \text{ for } \forall t \in [0, T].$$

Then we have

$$\begin{aligned} & J(u^\epsilon) - J(u) \\ = \; & E\int_0^T (-H(t, x(t), u^\epsilon(t), p(t), q(t))) - (-H(t, x(t), u(t), p(t), q(t)))dt + o(\epsilon) \end{aligned}$$

Proof of Necessary SMP

Proof is technical and lengthy. Essential steps are that for given optimal 4-tuple $(\bar{x}, \bar{u}, \bar{p}, \bar{q})$, define a functional $\mathcal{H}^{\bar{u}} : L^4(S; \mathbb{R}^k) \to \mathbb{R}$ as following

$$\mathcal{H}^{\bar{u}}(u) = E \int_0^T -H(t, \bar{x}(t), u(t), \bar{p}(t), \bar{q}(t))dt.$$

- To characterize Clarke's generalized gradient of J and $\mathcal{H}^{\bar{u}}$ at $\bar{u}$ and explore their relation and optimality condition

$$0 \in \partial J(\bar{u}) + N_{\mathcal{U}_{ad}}(\bar{u}) = \partial \mathcal{H}^{\bar{u}}(\bar{u}) + N_{\mathcal{U}_{ad}}(\bar{u}). \tag{23}$$

- To characterize Clarke's tangent cone $T_{\mathcal{U}_{ad}}(\bar{u})$ in $L^4(S; \mathbb{R}^k)$ space.
- To characterize (23) as that $\exists \zeta \in L^{\frac{4}{3}}(S; \mathbb{R}^k)$ such that

$$\begin{cases} E \int_0^T \langle \zeta(t), v(t) \rangle dt \leq 0 \; \textit{for} \; \forall v \in L^4(S; \mathbb{R}^k) \; \textit{such that} \\ v(t) \in T_U(\bar{u}(t)) \quad \textit{for every} \; t \in [0, T], \mathbb{P}\textit{-almost surely} \\ (\mathcal{H}^{\bar{u}})^o(\bar{u}; v) + E \int_0^T \langle \zeta(t), v(t) \rangle dt \geq 0 \; \textit{for} \; \forall v \in L^4(S; \mathbb{R}^k). \end{cases} \tag{24}$$

- To characterize pointwise conditions using measurable selection theorem.

Proof of Sufficient SMP

Given admissible pair (x, u), define

$$H(t, x(t), u(t)) \triangleq H(t, x(t), u(t), \bar{p}(t), \bar{q}(t)) \ \textit{for} \ \forall t \in [0, T], \ \mathbb{P}\text{-}a.s.$$

Under convexity condition, Clarke's generalized gradient and normal cone coincide with subdifferential and normal cone in sense of convex analysis. Moreover, combining (6) and concavity of $H(t, \bar{x}(t), \cdot)$ for all $t \in [0, T]$ a.s, we conclude that

$$H(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} H(t, \bar{x}(t), u), \ \ a.e. \ t \in [0, T], \ \mathbb{P}\text{-}a.s.$$

Define $\xi(t) \triangleq x(t) - \bar{x}(t)$ satisfying

$$\begin{cases} d\xi(t) &= \{b(t, x(t), u(t)) - b(t, \bar{x}(t), \bar{u}(t))\} \, dt \\ & \quad + \displaystyle\sum_{j=1}^{m} \left\{\sigma^j(t, x(t), u(t)) - \sigma^j(t, \bar{x}(t), \bar{u}(t))\right\} dW^j(t), \ t \in [0, T], \\ \xi(0) &= 0. \end{cases}$$

Following a standard separating hyperplane argument in convex analysis, Rockafellar (1973), Chapter 5, we obtain

$$\int_0^T \{H(t, x(t), u(t)) - H(t, \bar{x}(t), \bar{u}(t))\} \leq \int_0^T \langle H_x(t, \bar{x}(t), \bar{u}(t)), \xi(t) \rangle dt \tag{25}$$

for any admissible pair (x, u).

Applying Itô's formula to $\langle \bar{p}(t), \xi(t) \rangle$, noting convexity of h, inequality (25) and definition of Hamilitonian (4), we have

$$
\begin{aligned}
& E\{h(x(T)) - h(\bar{x}(T))\} \\
\geq & E\langle h_x(\bar{x}(T)), \xi(T\rangle)\rangle \\
= & - E\langle \bar{p}(T), \xi(T)\rangle \\
= & E\int_0^T \{\langle H_x(t, \bar{x}(t), \bar{u}(t)), \xi(t)\rangle - \langle \bar{p}(t), b(t, x(t), u(t)) - b(t, \bar{x}(t), \bar{u}(t))\rangle \\
& - \sum_{j=1}^m \langle \bar{q}_j(t), \sigma^j(t, x(t), u(t)) - \sigma^j(t, \bar{x}(t), \bar{u}(t))\rangle\} dt \\
\geq & - E\int_0^T \{f(t, x(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t))\} dt.
\end{aligned}
$$

Therefore $J(\bar{u}) \leq J(u)$ for all $u \in \mathcal{U}_{ad}$.

Conclusion

- We have proved a weak version of necessary and sufficient stochastic maximum principle.
- Instead of insisting on maximality condition of Hamiltonian, we showed that 0 belongs to sum of Clarke's generalized gradient of H and Clarke's normal cone at optimal control $\bar{u}$.
- Under certain concavity conditions on Hamiltonian and objective functions, necessary condition becomes sufficient.
- Weak SMP does not involve any second order terms, hence no second order differentiability of coefficients and objective functions is required, and hence considerably simplifies computation.
- Future research on this topic includes extension to more general stochastic control systems. We are currently working on these problems.